Separation of Perishable Inventories in Offline Retailing through Transshipment

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Abstract

Transshipment in retailing is a practice where one outlet ships its excess inventory to another outlet with inventory shortages. By balancing inventories, transshipment can reduce waste and increase fill rate at the same time. In this paper, we explore the idea of transshipping perishable goods with a fixed finite lifetime in offline grocery retailing. In the offline retailing of perishable goods, customers typically choose the newest items first, which can lead to substantial waste. We show that in this context, transshipment plays two roles. One is inventory balancing, which is well known in the literature. The other is inventory separation, which is new to the literature. That is, transshipment allows a retailer to put newer inventory in one outlet and older inventory in the other. This makes it easier to sell older inventory and reduces waste as a result. To understand how exactly inventories should be separated, we study an approximation in which computation relies on only two pieces of information, namely the number of items expiring in one period (old items) and that of the rest (new items). We show that the optimal policy can be characterized by two increasing switching curves. The two switching curves divide the entire state space into three regions. In the first region, only one outlet holds old items while both hold new items. In the second, one outlet holds old items and the other holds new items. In the third, only one outlet holds new items while both hold old items. We also conduct numerical studies to quantify the value of transshipment.

1 Introduction

The sale of perishable products accounts for over 50% of the business of grocery retailing (Laseter and Rabinovich 2012). As consumers become increasingly health conscious, the importance of perishable products can only grow. Besides, retailers also rely on perishable products to drive store traffic and gain competitive edge (Tsiros and Heilman 2005). Managing inventory of perishable products, however, is challenging and waste is substantial in retailing. Retailers remove items from the shelves when they are near or past their expiration dates. In America, approximately 40% of food produce is wasted, much of it in retailing (Kaye 2011). Waste in retailing is a problem in many other places too. According to a recent study by Friends of the Earth, the four main supermarkets in Hong Kong throw away 87 tons of food a day, and most of the waste ends up in landfill (Wei 2012). Many retailers have programs aimed at addressing the problem. For example, the Food Waste Reduction Alliance in the United States, the Waste and Resources Action Programme in the United Kingdom, and the Retailers' Environmental Action Programme in Europe were all established with waste reduction as their primary goal.

The main challenges in managing perishable products at offline grocery retailers are well known. First, demand is uncertain, and hence it is difficult to match supply with demand. Second, perishable items typically have very short lifetimes and hence they need to be sold in a short time window. Third, customers choose items on a last-in-first-out (LIFO) basis. Retailers replenish their shelves periodically. Whenever a new shipment arrives, the items on the shelves, which have shorter remaining lifetimes, are one step closer to the bin. Various ideas for managing perishable products have been adopted in practice and discussed in the academic literature. In particular, to induce more customers to choose older items first and hence leave fresher items on shelves, items on shelves can be arranged in a way that flesher items are harder to reach. But this practice can only make it harder for customers to retrieve the flesher items, not stop them from doing so. For the retailers with backroom storage, workers can stock shelves more frequently and in smaller quantity, effectively hiding fresher items from customers until older items are (almost) sold out. This tactic requires backroom refrigerated storage, constant monitoring of inventory levels on shelves and frequent replenishment. In this paper, we add a new weapon to the arsenal in the war against perishability: transshipment.

Transshipment has been widely studied in the operations management literature, but it has not been studied in retailing of perishable products under the LIFO rule. Existing research shows that its benefit comes from balancing inventories across different locations and hence reducing waste at some locations and shortages at others at the same time. In this study we explore the idea of transshipment in an offline retailer consisting of two outlets. The retailer replenishes its perishable products every period and at the end of each period, the retailer can either put the products that have not expired on clearance sale or carry them over to the next period. The products have a fixed finite lifetime. The retailer can also transship them from one outlet to the other. Our analysis shows that in this context, transshipment works very differently. It can balance inventories across different locations, similar to what is known in the literature. However, besides that, transshipment also plays a very different role in that it allows the separation of items with different remaining lifetimes. That is, transshipment allows the retailer to put newer inventory in one outlet and older inventory in the other. This makes it easier to sell older inventory and reduces waste as a result.

To understand how exactly inventories should be separated and how much benefit transshipment can generate, we consider an approximation. Under the approximation, the computation of optimal policy relies on only two pieces of information, namely the number of items expiring in one period (old items) and that of the rest (new items). We show that the optimal policy under the approximation can be characterized by two increasing switching curves. The two switching curves divide the entire state space into three regions. In the first region, only one outlet holds old items while both hold new items. In the second, one outlet holds old items and the other holds new inventory. In the third, only one outlet holds new items while both hold old items. Our numerical studies show that transshipment and clearance sales are substitutes in terms of *both* increasing profit and reducing waste. Transshipment can increase profit by as much as several percentage points. It is most valuable in increasing profit when the variable cost of products is high, outdating cost is high, clearance sale price is low, or demand variability is high.

To turn the idea of transshipment of perishables into reality, three important issues must be considered. First, the value of perishables per unit is typically low. Therefore, transshipment is viable economically only if the scale is large enough and the logistics is extremely efficient. Ideally, the transshipment should be integrated with the existing replenishment process so that the additional variable cost is minimal. Second, transshipment between outlets may require the cooperation of store managers whose incentive may not be aligned with that of the retailer. Third, our analysis suggests that in each period the retailer put old inventory in one outlet and new inventory in the other. The two outlets should take turn to be the one which receives the old inventory so that in the long run, the two outlets have equally fresh inventories. These issues can be difficult, but not impossible, barriers to overcome. How to overcome these barriers is beyond the scope of our current research. Instead, we focus on how transshipment should be implemented and its impact on profit. We believe this is the first step and only then will we know whether it is worth overcoming these barriers in implementation. Practically, our results are directly useful for retailers of perishable goods. Theoretically, the study also provides a completely new perspective on transshipment, an important concept in the field of operations management, and enriches the literature on perishable inventory and that on transshipment.

The remainder of the paper is organized as follows. In Section 2, we review related literature. We present the general model and its properties in Section 3. The general problem is computationally challenging. Therefore, we discuss its approximation in Section 4. The effects of transshipment on profit and waste are tested numerically in Section 5. We discuss an extension in Section 6 and conclude the paper in Section 7.

2 Related Literature

The study is directly related to two streams of literature in operations management. The first is the literature on transshipment. This literature is voluminous. The recent studies can be classified into two types. In the first type, there is a central decision maker who has access to full information and makes all the decisions. Representative studies of this type include, Abouee-Mehrizi et al. (2015) (lost sales), Hu et al. (2008) and Li and Yu (2014) (capacity constraints), and Yang and Qin (2007) (virtual transshipment). In all these studies, the objective is to characterize and compute the optimal replenishment and transshipment policies. In the second type of studies, there are multiple decision makers with different incentives. Various research questions have been raised. For example, Hu et al. (2007) focused on the question of whether a pair of coordinating transshipment prices, i.e., payments that each party has to make to the other for the transshipped goods, can be set globally such that the local decision makers are induced to make inventory and transshipment decisions that are globally optimal. Dong and Rudi (2004) and subsequently Zhang (2005) studied how transshipments affect independent manufacturers and retailers in a supply chain where retailers can transship inventory. Studies also exist that consider the cooperation and competition of retailers using cooperative game theory (e.g., Sosic 2006, Fang and Cho 2014). None of these studies has considered perishable products with a general lifetime.

The second stream is the literature on perishable inventory. A considerable renewed interest exists in the area (see, for example, Chao et al. 2018, Chao et al. 2015, Chen et al. 2014, Li and Yu 2014, and reviews by Karaesmen et al. 2011 and Nahmias 2011). Particularly related to our study is the strand of literature that considers the LIFO rule. Cohen and Pekelman (1978) analyzed the evolution over time of the age distribution of inventory. Under two particular order policies, constant order quantity and fixed critical number, they determined the shortages and outdates in each period by the age distribution and related them to inventory decisions. Pierskalla and Roach (1972) and Deniz et al. (2010) considered issuing endogenously and the set of feasible issuing rules includes LIFO. The former showed that under most of the objectives, first-in-first-out (FIFO) is the optimal issuing rule. The latter focused on finding heuristics to coordinate replenishment and issuing. Parlar et al. (2008) and Cohen and Pekelman (1979) compared FIFO issuance with LIFO issuance. None of the above-mentioned papers has considered the optimal inventory ordering policy under LIFO.

In spite of the practical relevance of the LIFO rule to retailing, little work has been done, especially in terms of optimal policies, perhaps due to the technical difficulties. However, recent progress is encouraging. Li et al. (2016) focused on the optimal policies on inventory control and clearance sales under LIFO and a general life time. They showed that a clearance sale may occur if the level of inventory with a remaining lifetime of one period is either very high or very low, a phenomenon that is unique to the LIFO rule. Furthermore, they showed that myopic heuristics requiring only information about total inventory and information about the inventory with a remaining lifetime of one period performed consistently well. Li et al. (2017) examined the impact of shelf-life-extending packaging on the optimal policy, cost, and waste. One interesting insight they gave was that although it may not be optimal in terms of cost, the adoption of shelf-life-extending packaging can consistently reduce waste substantially. None has considered transshipment in the literature on perishable inventory with a general lifetime.

The study closest to ours is perhaps that of Zhang et al. (2017). They studied transshipment of perishable inventory with a general lifetime between two locations. However, they assumed a FIFO rule and exogenous order-up-to levels, neither of which holds in offline grocery retailing. In summary, we are the first to consider transshipment of perishable inventory in offline grocery retailing.

3 The General Formulation

There are two identical outlets, indexed by superscript $i = 1, 2$, owned by the same retailer. The products they sell have an n-period lifetime. The products can be sold either at a regular price, p, or a clearance sale price, s. Under a regular price, the demand at each outlet is random and is modeled by random variable D^i. The demand under a clearance sale is so high (or s is so low) that inventory on clearance sales will never go unsold. More sophisticated pricing schemes have been used in services such as hotels and airlines, but are uncommon in offline retailing. We assume that D^1 and D^2 are identically distributed but not necessarily independent. The assumption is made so that we can sharpen the key insights and we will discuss the more general cases toward the end. Let $\Phi(\cdot)$, $\phi(\cdot)$, and μ denote the cumulative distribution function, the density function and mean for the demand, respectively.

The timing of events is as follows. 1) At the beginning of a period, the retailer determines how much to order and how much and what should be transshipped from one outlet to the other.

2) Then the random demand for regular sales is realized. 3) At the end of the period, the unsold inventory with a remaining lifetime of one period expires; and 4) the retailer determines how much of the inventory that has not expired should be carried over to the next period and how much should be put on clearance sale. Because there is no information updating between the ordering and transshipment decisions in 1) and the clearance sale decisions in 4), we redefine a period by moving 4) to the beginning of a period. In other words, all decisions are made at the beginning of a period. We assume that there is no transshipment cost in the model and the implication of transshipment cost will be discussed in the Conclusion section.

For outlet i, the initial inventory is represented by a vector $\mathbf{x}^i = (x_1^i, x_2^i, ..., x_{n-1}^i)$, where x_j^i represents the inventory with a remaining lifetime of j periods at outlet i. Let $x_j = x_j^1 + x_j^2$. The system state can be captured by $\mathbf{x} = (x_1, x_2, ..., x_{n-1})$. Let q^i be the order quantity of new items at outlet i. Let $\mathbf{z}^i = (z_1^i, .., z_{n-1}^i)$, where z_j^i is the inventory with a remaining life time of j periods that retail outlet i has after transshipment and clearance sale. As such, the total amount of inventory with a remaining lifetime of j periods available for regular sales is $z_j^1 + z_j^2$ and the amount sold in clearance sales is $x_j - z_j^1 - z_j^2$. Customers would always choose the freshest products first; that is, inventory leaves the retail shelf on a LIFO basis. Suppose the system state becomes $\mathbf{Y}^i(q^i, \mathbf{z}^i, D^i) = (Y_1^i, Y_2^i, ..., Y_{n-1}^i)$ in the next period. Then, for $1 \leq j \leq n-2$

$$Y_j^i(q^i, \mathbf{z}^i, D^i) = (z_{j+1}^i - (D^i - q^i - \sum_{k=j+2}^{n-1} z_k^i)^+)^+$$

and

$$Y_{n-1}^i(q^i, \mathbf{z}^i, D^i) = (q^i - D^i)^+.$$

The outdated amount is

$$S(q^i, \mathbf{z}^i, D^i) = (z_1^i - (D^i - q^i - \sum_{j=2}^{n-1} z_j^i)^+)^+.$$

Let c, θ, and α be the ordering cost, outdating cost, and the discounting factor, respectively. Without loss of generality, we assume that there is no holding cost. The dynamic programming formulation is as follows:

$$J_t(\mathbf{z}^i, q^i) = -s \sum_{j=1}^{n-1} z_j^i - cq^i + p\mathsf{E}\min(q^i + \sum_{j=1}^{n-1} z_j^i, D^i) - \theta \mathsf{E} S(q^i, \mathbf{z}^i, D^i), \tag{1}$$

and

$$v_t(\mathbf{x}) = s \sum_{j=1}^{n-1} x_j + \max\{J_t(\mathbf{z}^1, q^1) + J_t(\mathbf{z}^2, q^2) + \alpha \mathsf{E} v_{t+1}(\sum_{i=1}^{2} \mathbf{Y}^i(q^i, \mathbf{z}^i, D^i))\}, \tag{2}$$

subject to $z_j^1 + z_j^2 \leq x_j$, $z_j^i \geq 0$, $q^i \geq 0$ for all $i = 1, 2$ and $j = 1, 2, ..., n-1$. On the right-hand side of (1), the second term is the purchasing cost, the third term the revenue from regular sales, and the last term the outdating cost. The sum of the first terms on the right-hand sides of (1) and (2) represents the revenue from clearance sales. Hence $J_t(\mathbf{z}^i, q^i)$ is the one-period profit generated at outlet i. The planning horizon is T and the terminal condition is $v_{T+1}(\mathbf{x}) = s\sum_{i=1}^{n-1} x_i$. Denote by $(\bar{z}_j^i, \bar{q}^i)$, $j = 1, 2, ..., n-1$ and $i = 1, 2$, the optimal solution to (2). The optimal policy is time dependent. But to simplify notation, we omit the time index when there is no risk of confusion. We also assume that $s < c$ to avoid trivial solutions.

Let $\mathbf{e}_i$ denote an $n-1$ dimensional unit vector where the *i-th* element equals one and all other elements equal zero. Let δ be a small positive number. We can show the following results on the marginal values of initial inventories.

Lemma 1

(i) $v_t(\mathbf{x} + \delta\mathbf{e}_i) \leq v_t(\mathbf{x} + \delta\mathbf{e}_{i+1})$*;*

(ii) $s\delta \leq v_t(\mathbf{x} + \delta\mathbf{e}_i) - v_t(\mathbf{x}) \leq c\delta$*;*

(iii) $J_t(\mathbf{z}^i, q^i)$ *is submodular in* (z_1^i, q^i) *and* (z_1^i, z_j^i) *for* $j \geq 2$*.*

In the next theorem, we show that if items with a two-period or longer lifetime are sold through clearance sales under the optimal policy, then all the older inventories are cleared and no new items are ordered. In addition, if the total inventory on hand with a remaining lifetime of at least two periods is large enough, then at least one of the two outlets will not be keeping inventory with a remaining lifetime of one period. Let $l_0 = \Phi^{-1}(\frac{p-s}{p+\theta})$, which represents the optimal quantity of inventory with a remaining lifetime of one period an outlet should carry over to the next period when the selling price is p, disposal cost is θ, and the opportunity cost (clearance price) is s.

Theorem 1

(i) If $\bar{z}_i^1 + \bar{z}_i^2 < x_i$ *for some* $i \geq 2$*, then* $\bar{z}_j^1 = \bar{z}_j^2 = 0$ *for all* $j < i$ *and* $\bar{q}^1 = \bar{q}^2 = 0$*;*

(ii) If $\sum_{i=2}^{n-1} x_i \geq 2l_0$*, then either* $\bar{z}_1^1 = 0$ *or* $\bar{z}_1^2 = 0$*.*

In Part (ii) of Theorem 1, if the optimal solution is symmetrical, then both $\bar{z}_1^1$ and $\bar{z}_1^2$ should be zero. However, the optimal solution may not be symmetrical, even though the two outlets are identical and face identically distributed demands. Indeed, when the inventory is depleted on an FIFO basis, we can show that there is a symmetrical optimal solution. However, this is not the case in our setting.

The following result requires the random demands to follow a PF_2 distribution. PF_2 distributions are also known to have log-concave densities (Ross 1983). This is a common assumption in the inventory literature (e.g., Huggins and Olsen 2010, Li et al. 2016), and the class of distributions includes many commonly used distributions. PF_2 distributions have the following smoothing property: if D is a PF_2 random variable and $f(x)$ is quasiconcave, then $\mathsf{E}f(x-D)$ is quasiconcave.

Theorem 2 *Suppose that D^i has a PF_2 distribution. If $\bar{z}_i^1 = \bar{z}_i^2$ for $2 \leq i \leq n-1$, then there is an optimal policy such that at least one of $\bar{z}_1^1$, $\bar{z}_1^2$, $\bar{q}^1$ and $\bar{q}^2$ is zero.*

In the above theorem and the following Theorems 3 to 9, we show that there is an optimal policy that possesses certain properties. If the cumulative distribution functions of the demands are strictly increasing, then we can show that all optimal polices possess the properties. Theorem 2 includes two special cases. The first case is when $x_i = 0$ for all $i = 2, ..., n-1$, and the second is when the lifetime $n = 2$. In both cases, the condition $\bar{z}_i^1 = \bar{z}_i^2$ for $2 \leq i \leq n-1$ is obviously satisfied. Transshipment allows the retailer to send the oldest inventory to one outlet and the newest inventory to the other (i.e., separation of inventories) as well as send inventory from the outlet with excess inventory to the one with shortage (i.e., balance of inventories). When exactly one of $\bar{z}_1^1$, $\bar{z}_1^2$, $\bar{q}^1$ and $\bar{q}^2$ is zero, or when exactly two of them are zero and one outlet holds only the oldest inventory and the other holds only the newest inventory, separation of inventories occurs. The following theorem looks at the similar issues from a different angle.

Theorem 3

(i) If $\bar{z}_1^1 > 0$ and $\bar{z}_1^2 > 0$, then $\sum_{j=1}^{n-1} \bar{z}_j^1 + \bar{q}^1 = \sum_{j=1}^{n-1} \bar{z}_j^2 + \bar{q}^2$;

(ii) If $\bar{z}_1^1 > 0$ and $\bar{z}_1^2 = 0$, then $\sum_{j=1}^{n-1} \bar{z}_j^1 + \bar{q}^1 \leq \sum_{j=1}^{n-1} \bar{z}_j^2 + \bar{q}^2$.

According to Theorem 3 (i), if both outlets hold the oldest inventory, then inventory is balanced in the sense that the two outlets hold the same amount of total inventory. In light of Part (ii) of Theorem 1, both outlets hold the oldest inventory when the total inventory with a remaining lifetime of two periods or longer is not too large. However, although the two outlets hold the same amount of inventory, the types of inventory they hold may be different and each may hold a certain type of inventory that the other does not hold. In these situations, separation of inventories occur. For example, when $n = 2$, based on Theorem 2, either exactly one of $\bar{q}^1$ and $\bar{q}^2$ is zero (separation of inventories), or both $\bar{q}^1$ and $\bar{q}^2$ are zero and $\bar{z}_1^1 = \bar{z}_1^2$ (balance of inventories). Part (ii) of the theorem is because the higher the total inventory with a remaining lifetime of two periods or longer, the higher the expected outdating. As a special case of part (ii), when $n = 2$, if $\bar{z}_1^1 > 0$ and $\bar{z}_1^2 = 0$, then $\bar{z}_1^1 + \bar{q}^1 \leq \bar{q}^2$ and hence $\bar{q}^1 < \bar{q}^2$. Furthermore, when $n = 2$, we can show that if $\bar{z}_1^1 = \bar{z}_1^2 = 0$, then $\bar{q}_1 = \bar{q}_2$.

4 Approximations

The structural properties in Section 3 provide useful guidance. However, to put the ideas into practice, there are still open questions. First, how much should each outlet order in each period, and how much existing inventories should be sold in clearance sales and how much should be carried over to the next period? Second, how should the inventories be allocated between the two outlets? Under what conditions should inventories be separated and what conditions they be balanced? Third, what would be the impact of transshipment on profit and waste? To answer these questions with the general formulation in Section 3, we need to know how many units of inventory there are in each age group, and with that information, to solve a dynamic program with a multi-dimensional state space and a non-concave objective function. The former is impossible given the current bar code design and standard and the latter is challenging computationally. Approximation is the only way forward.

4.1 An Approximation and Its Theoretical Bounds

To better understand the complex problem, we first look at a very special case where all demands are known in advance and the discount factor is 1. In this case, the optimal policy has a simple form, and most importantly, the states can be aggregated. Let $x_{[2]} = \sum_{j=2}^{n-1} x_j$. The result is summarized in the following lemma.

Lemma 2 *Suppose that $\alpha = 1$ and all demands are known. Let the demand at outlet i in period t be d_t^i. Then for any initial inventory, the optimal policy in period t has the following form:*

(i) If $x_1 + x_{[2]} < d_t^1 + d_t^2$, then there are no clearance sales; outlet i holds d_t^i units of total inventory after replenishment and transshipment.

(ii) If $x_1 + x_{[2]} \geq d_t^1 + d_t^2$ and $x_{[2]} < d_t^1 + d_t^2$, then there is no replenishment; clearance only takes place for the oldest inventory; outlet i holds d_t^i units of total inventory after clearance and transshipment.

(iii) If $x_1+x_{[2]} \geq d_t^1+d_t^2$ and $x_{[2]} \geq d_t^1+d_t^2$; there is no replenishment and all oldest inventories are cleared; allocate, from the oldest to the newest, just enough inventories to the outlet with a larger demand in period t to meet its demand and allocate the rest to the other outlet.

Under the conditions in Lemma 2, the computation of the optimal policy depends on the states only through x_1 and $x_{[2]}$. In addition, the allocation of inventories that are carried over to the next period across the two outlets should maximize the "separation" - older inventories to the larger outlet and newer to the other. Motivated by Lemma 2, we are looking for approximations that depend only on x_1 and $x_{[2]}$ when computing the policy. One simple way to accomplish that is to approximate the profit-to-go by a linear function. That is, in period t, we let $v_{t+1}(\mathbf{x}) = v\sum_{j=1}^{n-1} x_j$, where v is a number bounded by c and s (e.g., $v = (s+c)/2$) because the marginal value of inventory is bounded by c and s. Under such an approximation, we assign the same salvage value v to all the inventories with a remaining lifetime of two periods or longer.

In this section, for ease of exposition, we call x_1 the *old inventory* and $x_{[2]}$ the *new inventory*. We use z_1^i and $z_{[2]}^i$ to represent the amount of old inventory and new inventory, respectively, allocated to outlet i for regular sales. Let y^i be the amount of new inventory after ordering at outlet i. That is, y^i is the order-up-to level for new inventory at outlet i. To avoid the need for additional notation, we continue to use J to represent the one-period profit for an outlet when the above approximation is used. Let

$$J(z_1, y) = -sz_1 - cy + p\mathsf{E}\min(D, z_1+y) - \theta\mathsf{E}(z_1-(D-y)^+)^+ + \alpha v\mathsf{E}(y-D)^+,$$

and, to find a heuristic policy, we solve the following one-period optimization problem:

$$\max\{(c-s)(z_{[2]}^1 + z_{[2]}^2) + J(z_1^1, y^1) + J(z_1^2, y^2)\} \tag{3}$$

subject to $z_1^1 + z_1^2 \leq x_1$, $z_{[2]}^1 + z_{[2]}^2 \leq x_{[2]}$, $z_1^i \geq 0$, $z_{[2]}^i \geq 0$, $y^i \geq z_{[2]}^i$ for $i = 1, 2$. For period T, we choose $v = s$. For any other period, we choose a $v \in [s, c]$ such that the total expected profit for the entire planning horizon under the approximation is the highest. Denote by $(z_1^{i,H}, z_{[2]}^{i,H}, y^{i,H})$, $i = 1, 2$ the optimal solution to the optimization problem (3).

Denote by $(\mathbf{z}^{i,H}, q^{i,H})$, $i = 1, 2$, the decisions under the approximation. Here $\mathbf{z}^{i,H} = (z_1^{i,H}, z_2^{i,H}, ..., z_{n-1}^{i,H})$. Clearly, $q^{i,H} = y^{i,H} - z_{[2]}^{i,H}$. By computing (3), we have computed $z_1^{i,H}$, $q^{i,H}$, $z_{[2]}^{i,H}$, where $z_{[2]}^{i,H} = \sum_{j=2}^{n-1} z_j^{i,H}$. For implementation, we need to "de-aggregate" $z_{[2]}^{i,H}$; that is, we need to determine the allocation of the new inventory across the two outlets such that

$\sum_{j=2}^{n-1} z_j^{i,H} = z_{[2]}^{i,H}$ for $i \in \{1,2\}$ and $\sum_{i=1}^{2} z_j^{i,H} \leq x_j$ for $j = 2,3,...,n-1$. The way inventory is allocated affects the initial inventory in the next period and hence the profit and waste.

Motivated by the earlier analytical results, we adopt the following sequential method to allocate new inventory to maximize the "separation". Before allocation, an amount of new inventory equal to $\sum_{i=2}^{n-1} x_i - \sum_{i=1}^{2} z_{[2]}^{i,H}$ is first cleared, obviously, from older inventories to newer ones. The amount of new inventory that is carried over the next period and needs to be allocated across the two outlets is $\sum_{i=1}^{2} z_{[2]}^{i,H}$. The "demand" for the new inventory in outlet i is $(D^i - q^{i,H})^+$. Therefore, the outlet with a smaller $q^{i,H}$ can be treated as a "larger" outlet. Suppose outlet i is the larger outlet. We allocate the inventories, from oldest (i.e., x_2) to the newest (i.e., x_{n-1}) , to outlet i using a greedy algorithm until the total allocation equals $z_{[2]}^{i,H}$, and then allocate the rest to the other outlet. The following Figure 1 illustrates the method. In the figure, the unshaded area represents the cleared inventories and the shaded area the inventories for regular sales. In (a), none of the new inventories is cleared and in (b), some of the new inventories are cleared.

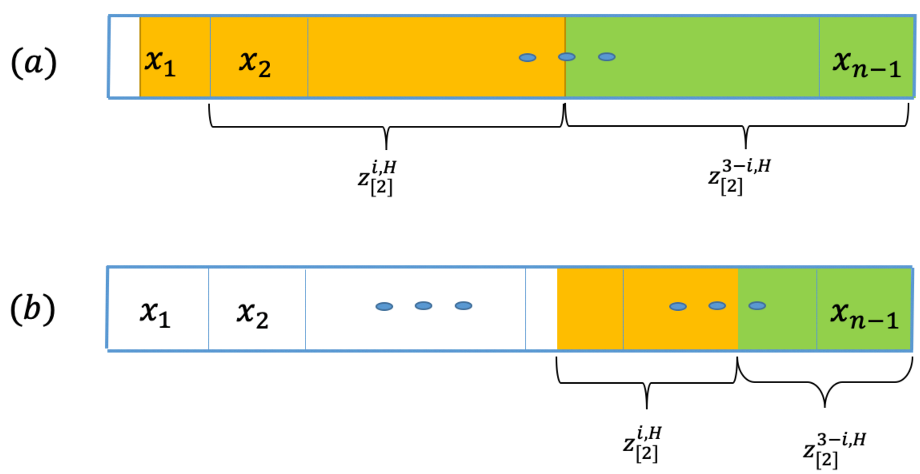

Figure 1: Allocation of the new inventory when i is the larger outlet

We use two examples to illustrate the allocation method. In both examples, we assume $n = 6$ and $(x_1, x_2, .., x_5) = (5,1,1,1,2)$. In the first example, suppose that the optimal solution for the approximation model is $(z_1^{1,H}, z_{[2]}^{1,H}, y^{1,H}) = (0,2,4)$ and $(z_1^{2,H}, z_{[2]}^{2,H}, y^{2,H}) = (0,3,3)$. In this case, $(q^{1,H}, q^{2,H}) = (2,0)$ and hence outlet 2 is the larger outlet. The inventory composition after allocation at outlets 1 and 2 are $(0,0,0,0,2)$ and $(0,1,1,1,0)$, respectively. In the second example, suppose that the optimal solution for the approximation model is $(z_1^{1,H}, z_{[2]}^{1,H}, y^{1,H}) =$

$(2, 2, 2)$ and $(z_1^{2,H}, z_{[2]}^{2,H}, y^{2,H}) = (0, 3, 4)$. In this case, $(q^{1,H}, q^{2,H}) = (0, 1)$ and hence outlet 1 is the larger outlet. The inventory composition after allocation at outlets 1 and 2 are $(2, 1, 1, 0, 0)$ and $(0, 0, 0, 1, 2)$, respectively.

The above approximation, which includes the computation in (3) and the allocation of new inventory across the outlets, is optimal under the conditions in Lemma 2. It is appealing for the following reasons. First, the computation of (3) requires only two pieces of information - the oldest inventory and the total inventory. Retailers typically check and remove expired items manually. The process of putting items on clearance sales is also manual. The information about x_1 can be obtained during these manual processes. Second, the properties related to the original problem in Section 3 continue to hold true in the approximation model. Specifically, the marginal value of inventory is bounded by c and s, newer inventory is more valuable than older inventory (Lemma 1), older inventory should be cleared before newer inventory, and ordering occurs only when inventory with a remaining lifetime equal to or longer than two periods is not sold in clearance sales (Theorem 1). In the original model we have shown that the retailer may separate the oldest inventories from the newest ones through transshipment. In the approximation model (3), the incentive to separate inventories remain and in the algorithm for allocating new inventories, we maximize the separation of inventories. Third, it allows us to fully characterize the optimal solution *analytically*. Not only does it give us a good tool to compute the optimal policy approximately, it also enhances our understanding of separation of inventories, a phenomenon new to the field.

In what follows, we show that the loss of expected profit from using the approximation relative to the maximal profit has an easily computable upper bound. Let's denote the total expected profit from period t to the end of the horizon when the approximation is used as $v_t^H(\mathbf{x})$. Let

$$o(x) = s\mathsf{E}(x - \sum_{j=1}^{T}\sum_{i=1}^{2} D_j^i)^+ + c\sum_{t=1}^{T}\alpha^{t-1}\mathsf{E}\min\{(x - \sum_{j=1}^{t-1}\sum_{i=1}^{2} D_j^i)^+, \sum_{i=1}^{2} D_t^i\},$$

which measures the expected value of x units of initial inventory if the inventory is non-perishable. Here, the first term of on the right hand side represents the revenue from clearance sales and the second term represents the savings in purchasing cost. Let $r(x_1, x_{[2]})$ denote the optimal value of the optimization problem (3) when we choose $v = s$. Then the gap between the performance of the approximation and the optimal has a simple upper bound.

Theorem 4 *Assume that* $\alpha < 1$*. Then*

$$\frac{v_1(\mathbf{x}) - v_1^H(\mathbf{x})}{v_1(\mathbf{x})} \le 1 - \frac{s\sum_{j=1}^{n-1} x_j + r(x_1, \sum_{i=2}^{n-1} x_i) + \frac{\alpha(1-\alpha^{T-1})}{1-\alpha} r(0,0)}{\frac{1-\alpha^T}{1-\alpha} 2(p-c)\mu + o(\sum_{i=1}^{n-1} x_i)}.$$

(i) Suppose $\mathbf{x} = \mathbf{0}$*,*

$$\frac{v_1(\mathbf{0}) - v_1^H(\mathbf{0})}{v_1(\mathbf{0})} \le \frac{(p-c)\mu - r(0,0)/2}{(p-c)\mu}.$$

(ii)

$$\lim_{\alpha \to 1} \frac{v_1(\mathbf{x}) - v_1^H(\mathbf{x})}{v_1(\mathbf{x})} \le \frac{(p-c)\mu - r(0,0)/2}{(p-c)\mu + \lim_{\alpha \to 1} o(\sum_{i=1}^{n-1} x_i)/T}.$$

(iii) Suppose the demand is compound Poisson, where λ *is the arrival rate of customers and customers purchase batch sizes that are independent and identically distributed with finite second moment. Then*

$$\lim_{\lambda \to \infty} \frac{v_1(\mathbf{x}) - v_1^H(\mathbf{x})}{v_1(\mathbf{x})} = 0.$$

To compute the upper bound, we need to compute the function $r(\cdot, \cdot)$, which requires no dynamic recursion and is easy. In part (i), because

$$\frac{r(0,0)}{2} = \max_{y \ge 0}\{-cy + p\mathsf{E}\min\{D^i, y\} + \alpha s \mathsf{E}(y - D^i)^+\},$$

$(p-c)\mu - r(0,0)/2$ is the value of information in the newsvendor problem - how much better we can do if demand is known before the ordering decision is made. For a general initial state $\mathbf{x}$, part (ii) provides a bound that accounts for the expected value of initial inventory when the discount factor is close to 1. As T approaches infinity, $\lim_{\alpha \to 1} o(\sum_{i=1}^{n-1} x_i)/T$ becomes zero and so the bound is also equal to the same value of information. In this case, both $v_1(\mathbf{x})$ and $v_1^H(\mathbf{x})$ approach infinity when T does so. Although $v_1(\mathbf{x})$ diverges faster than $v_1^H(\mathbf{x})$, the rates at which they diverge are of the same order. In both (i) and (ii), when the demand variance is zero, the bound is zero, consistent with Lemma 2. Part (iii) shows that the approximation is asymptotically optimal as the demand rate grows large. Under compound Poisson demand, as the arrival rate approaches infinity, both the approximation and the optimal policy converge to a policy that orders up to the mean demand in each period. Similar asymptotic results have appeared in Zhang et al. (2020) and Gallego and van Ryzin (1994).

4.2 The Properties of the Approximation

As building blocks for characterization of the optimal actions under the approximation, we can show that $J(z_1, y)$ is submodular in (z_1, y), which is similar to Lemma 1 (iii), and if D has a

PF_2 distribution, then $J(z_1, y)$ is quasiconcave in y and hence the computation of the optimal solution is easy. These technical properties are summarized in Lemma 3 in the Appendix.

The following theorem is a stronger version of Theorems 2 and 3.

Theorem 5 *There is an optimal policy of (3) such that at least one of the two outlets holds either the old inventory or the new inventory, but not both types of inventory; that is, at least one of $z_1^{1,H}, z_1^{2,H}, y^{1,H}$ and $y^{2,H}$ is zero. In particular, there are three possibilities:*

(i) $z_1^{1,H} = z_1^{2,H} = 0$. In this case, $y^{1,H} = y^{2,H}$;

(ii) $z_1^{1,H} > 0, z_1^{2,H} > 0$. In this case, $z_1^{1,H} + y^{1,H} = z_1^{2,H} + y^{2,H}$ and at least one of $y^{1,H}$ and $y^{2,H}$ is zero;

(iii) Either $z_1^{1,H} > 0$ or $z_1^{2,H} > 0$ but not both.

As we mentioned earlier, the retailer may use transshipment to separate or balance inventories, depending on the context. In case (i) above, the solution is symmetric and the inventories are balanced. In case (ii), both separation and balance of inventories occur. All the new inventory is sent to one outlet, but the two outlets hold the same total inventory. In case (iii), all the old inventory is sent to one outlet and inventories are separated. In the following several theorems, we will provide specific conditions, in terms of the state variables, under which each case will occur.

The optimal separation policy is given in the next theorem.

Theorem 6 *There exist two increasing functions $B(x_1) \geq C(x_1)$ such that*

(i) If $x_{[2]} > B(x_1)$, then at most one outlet holds old inventory while both outlets hold new inventory;

(ii) If $x_{[2]} < C(x_1)$, then at most one outlet holds new inventory while both outlets hold old inventory;

(iii) If $C(x_1) \leq x_{[2]} \leq B(x_1)$, then one outlet holds old inventory while the other holds new inventory.

Theorem 6 provides an optimal mapping between the allocation of inventories and the state. It reconfirms that it is suboptimal to stock both old and new inventories in both outlets. In addition, in light of Theorem 5, when $x_{[2]} < C(x_1)$, the two outlets hold the same amount of total inventory. We further characterize the structure of the optimal clearance sales policy of old items in the following theorem.

Theorem 7

(i) For $x_{[2]} < C(x_1)$, the old inventory is put on clearance sales if and only if $x_{[2]} \geq 2l_0 - x_1$.

(ii) For $C(x_1) \leq x_{[2]} < B(x_1)$, the old inventory is put on clearance sales if and only if $x_1 \geq l_0$.

(iii) For $x_{[2]} > B(x_1)$, there exists a function $e_0(x_1)$ such that the old inventory is put on clearance sales if and only if $x_{[2]} \geq e_0(x_1)$.

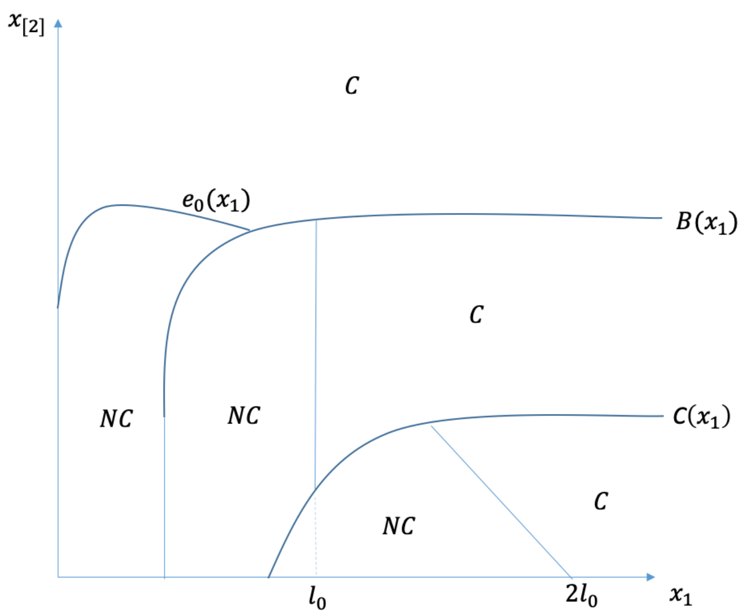

Figure 2: Separation of Inventories and Clearance Sales (Note: Clearance and Non-clearance Regions are Labeled by "C" and "NC")

The results in Theorems 6 and 7 are shown in Figure 2. The clearance and non-clearance regions are labeled by "C" and "NC", respectively. In Figure 2, in the region between $C(x_1)$ and $B(x_1)$, one outlet holds old inventory and the other holds new inventory; that is, old and new inventories are separated. In this case, if x_1 is greater than l_0, then x_1 is cleared down to l_0; otherwise, there is no clearance sale of x_1. This is consistent with Theorem 5 where we show that inventories are separated if and only if there is old inventory in the system after clearance sales. In addition, we also observe from Figure 2 that balance of inventories in Theorem 5(i) can only take place in the region where $x_{[2]} > B(x_1)$.

Let $u = \arg\max_{y \geq 0} J(0, y)$. Then, $u = \Phi^{-1}(\frac{p-c}{p-\alpha v})$ and it represents the optimal ordering quantity of new items when the initial inventory of old items is zero. The structure of the optimal ordering policy is given in the following theorem.

Theorem 8

(i) If $x_{[2]} < C(x_1)$, there exists a function $e_1(x_{[2]}) \leq 2l_0 - x_1$ such that new items are ordered if and only if $x_1 \leq e_1(x_{[2]})$.

(ii) If $C(x_1) \leq x_{[2]} < B(x_1)$, new items are ordered if and only if $x_{[2]} \leq u$.

(iii) If $x_{[2]} > B(x_1)$, there exists a decreasing function $e_2(x_1)$ such that new items are ordered if and only if $x_{[2]} \leq e_2(x_1)$.

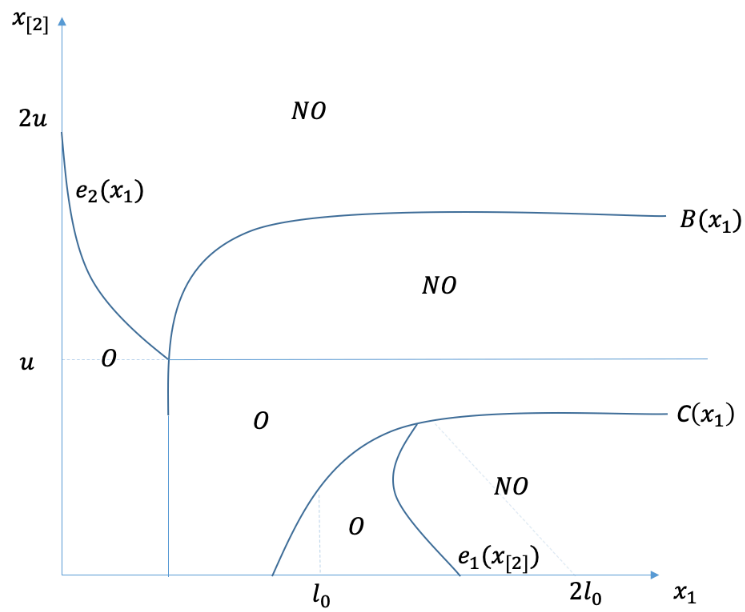

Figure 3: Optimal Policy on Ordering (Note: Ordering and Non-ordering Regions are Labeled by "O" and "NO")

The results in Theorem 8 can be visualized in Figure 3. The ordering and non-ordering regions are labeled by "O" and "NO", respectively. The optimal ordering quantity is monotonically decreasing if the inventory level of old items increases, but not necessarily decreasing when there are more new items. Similar results have been established by Li et al. (2016). Technically, this happens because even though $J(z_1^i, y^i)$ is quasiconcave in y^i, $\max_{z_1^i \geq 0} J(z_1^i, y^i)$ is not necessarily quasiconcave in y^i.

5 Numerical Studies

In this section, through numerical studies, we first test the performance gap between the approximation and the optimal policy. We then investigate two related issues. First, clearance sales and transshipment are two tools for retailers to fight perishability, but how are they related in creating value in terms of increasing profit and reducing waste? Second, under what circumstances does transshipment create the most value in increasing profit? For all the numerical studies, when the lifetime is no greater than 3, we compute the optimal policy for the dynamic programming formulation in Section 3. For any longer lifetimes, we compute the optimal policy for the approximated formulation in Section 4.

5.1 The Performance of the Approximation

In this subsection, we investigate the performance of the approximation that we proposed in Section 4. Earlier we established an upper bound for the performance gap between the approximation and the optimal policy, but the actual performance of the approximation is much better than what the bound would suggest. We also compare its performance with that of a quadratic approximation of the profit-to-go function, i.e., $v_{t+1}(\mathbf{x}) = \nu_1(\sum_{j=1}^{n-1} x_j)^2 + \nu_0(\sum_{j=1}^{n-1} x_j)$. We set lifetime $n = 3$. The performance of approximation is given by $\frac{v_1(\mathbf{0})-v_1^A(\mathbf{0})}{v_1(\mathbf{0})} \cdot 100\%$, where $v_1^A(\mathbf{0})$ is the total profit at the beginning of the planning horizon with initial state $\mathbf{x} = \mathbf{0}$ under one of the two approximations. To calculate $v_1^A(\mathbf{0})$, we search for the optimal v, ν_0 and ν_1 for each approximation that lead to the maximum value of $v_1^A(\mathbf{0})$.

The numerical results are summarized in Table 1. The performance of the approximation is surprisingly good and is close to be optimal when the margin of regular sales is not too low. The good performance of the approximation is probably because the approximation retains all the important analytical properties of the original dynamic program, and in spite of its simplicity, the approximation has already utilized the most crucial information - the oldest inventory and the total inventory. The quadratic approximation of the profit-to-go is better than the linear approximation, but only slightly.

5.2 Value of Transshipment and Clearance Sales

In this subsection, we study the value of transshipment and clearance sales in terms of both improving profit and reducing waste. We let the lifetime be 3 and calculate the profit and waste under the optimal policy of the dynamic program at initial state $\mathbf{x} = \mathbf{0}$ for four different

Table 1: The performance of approximations under linear and quadratic approximations of the profit-to-go function (%)

			$c=4$				$c=8$			
			$s=0$	$s=1$	$s=2$	$s=3$	$s=0$	$s=2$	$s=4$	$s=6$
$\sigma=1$	Linear	$\theta=0$	0.12	0.12	0.35	0.38	1.23	1.23	1.11	1.02
		$\theta=2$	0.40	0.44	0.94	0.42	1.28	1.28	1.37	0.95
		$\theta=4$	0.45	0.57	0.76	0.38	1.51	1.51	1.37	1.01
		$\theta=6$	0.75	0.42	0.60	0.38	1.18	1.18	1.26	1.04
	Quadratic	$\theta=0$	0.07	0.12	0.20	0.32	1.10	1.08	1.11	1.02
		$\theta=2$	0.40	0.31	0.80	0.33	0.89	1.03	1.35	0.95
		$\theta=4$	0.45	0.55	0.61	0.38	1.41	1.07	1.29	1.01
		$\theta=6$	0.70	0.40	0.60	0.38	1.13	1.13	0.98	1.02
$\sigma=2$	Linear	$\theta=0$	0.32	0.32	0.31	0.52	1.82	1.94	2.11	1.79
		$\theta=2$	0.33	0.28	0.78	0.31	2.20	2.37	1.97	1.45
		$\theta=4$	0.53	0.48	0.73	0.28	1.43	1.94	2.15	1.36
		$\theta=6$	0.80	0.30	0.66	0.34	1.65	2.22	2.55	1.42
	Quadratic	$\theta=0$	0.22	0.20	0.29	0.44	1.70	1.82	2.00	1.76
		$\theta=2$	0.21	0.16	0.74	0.31	2.20	2.17	1.96	1.45
		$\theta=4$	0.45	0.48	0.70	0.28	1.03	1.91	2.05	1.36
		$\theta=6$	0.62	0.30	0.63	0.28	1.61	2.01	2.31	1.36

Note. $T=10$, $n=3$, $p=10$, $\alpha=0.99$, $D=Normal(3,\sigma)$ *truncated below by 0.*

scenarios depending on whether or not transshipment and/or clearance sales are adopted. The results are presented in Table 2.

Table 2: Value of transshipment and clearance sales.

$c = 3, \theta = 1, s = 1$		Transshipment		
Profit		Y	N	Diff
Clearance	Y	175.80	175.20	0.60
	N	175.22	174.16	1.06
	Diff	0.58	1.04	
		Transshipment		
Waste		Y	N	Diff
Clearance	Y	2.66	2.68	-0.02
	N	4.26	4.37	-0.11
	Diff	-1.60	-1.69	

$c = 7, \theta = 1, s = 1$		Transshipment		
Profit		Y	N	Diff
Clearance	Y	53.00	51.67	1.33
	N	52.86	51.34	1.52
	Diff	0.14	0.33	
		Transshipment		
Waste		Y	N	Diff
Clearance	Y	1.98	2.27	-0.29
	N	1.98	2.34	-0.36
	Diff	0.00	-0.10	

$c = 3, \theta = 2, s = 1$		Transshipment		
Profit		Y	N	Diff
Clearance	Y	174.39	173.94	0.45
	N	170.86	169.93	0.93
	Diff	3.53	4.01	
		Transshipment		
Waste		Y	N	Diff
Clearance	Y	0.76	0.78	-0.02
	N	4.09	7.81	-3.72
	Diff	-3.33	-7.03	

$c = 7, \theta = 2, s = 4$		Transshipment		
Profit		Y	N	Diff
Clearance	Y	54.36	53.91	0.45
	N	53.33	51.26	2.07
	Diff	1.03	1.65	
		Transshipment		
Waste		Y	N	Diff
Clearance	Y	0.99	0.65	0.34
	N	1.59	4.42	-2.83
	Diff	-0.60	-3.77	

Note. $T = 15$, $n = 3$, $p = 10$, $\alpha = 0.99$, $D = Uniform[0, 2]$.

There are two observations from the table. First, clearance sales and transshipment are substitutes in both improving profit and reducing waste; i.e. having clearance sales decreases the value of having transshipment in improving profit or reducing waste. Second, in the presence of clearance sales, transshipment reduces waste in most cases, but not always. Exceptions happen for example when $c = 7$, $\theta = 2$, and $s = 4$. Since transshipment may reduce the need for clearance sales, the outlets may carry more old inventory to future periods with transshipment. In addition, the retailer may order more when transshipment is used. Both may increase waste.

5.3 The Impact of Transshipment on Profit

For $n \leq 3$, we can calculate the difference in profit between the optimal policy of the dynamic program with transshipment and that without. For $n \geq 4$, which might be more realistic, computing the optimal policies of the dynamic program is no longer possible. Instead, we compare the performance of the approximation with transshipment against that of the same approximation but without transshipment (i.e., the two outlets are managed independently using the same myopic policies). We consider a 15-period horizon, i.e. $T = 15$. The values taken by system parameters are: $s \in \{0, 1, 2, 3\}$ when $c = 4$, $s \in \{0, 2, 4, 6\}$ when $c = 8$, and $\theta \in \{0, 2, 4, 6\}$. The demand distribution is assumed to be normal with mean $\mu = 3$, standard deviation $\sigma \in \{1, 2\}$, and truncated below by zero. The discount rate $\alpha = 0.99$, and the regular sales price r is held constant at 10.

We summarize the numerical results in Table 3. In these studies, the increase in profit as a result of transshipment can be more than 4%, which is nontrivial given that grocery retailing is a low-margin business. Transshipment can increase profit the most when the ordering cost c is high, clearance sale price s is low, the outdating cost θ is high, and demand variability is high. When c is high or s is low, using clearance sale strategy to clear inventories approaching their expiration dates is costly. As a result, transshipment can have a greater impact. When θ increases, the outdating cost eats into the retailer's profit in a more significant way. Transshipment, which can reduce waste if other things being equal, again has a greater impact. Finally, the effect of variability is consistent with what we know about pooling, of which transshipment is an example.

6 Non-identically Distributed Demands

In the earlier analysis, to avoid the obfuscation of main insights, we have assumed that the two outlets face identically distributed demands. Now let us consider a more realistic case where one outlet faces a stochastically larger demand than the other. Without loss of generality, suppose D^2 is stochastically larger than D^1 (Ross 1983); that is, $D^2 \geq_{st} D^1$. According to our analysis, it is not a good idea to put old and new inventory in the same outlet. When the demands are not identically distributed, a new issue arises: which outlet should get the new inventory and which outlet should have the old inventory? On the one hand, it makes sense to allocate the old inventory to outlet 2 because its stochastically larger demand makes it more likely to deplete the old inventory. On the other hand, outlet 2 requires a higher total inventory to fill its larger

Table 3: Percentage Increase in Profit as a Result of Transshipment.

			$c=4$				$c=8$			
			$s=0$	$s=1$	$s=2$	$s=3$	$s=0$	$s=2$	$s=4$	$s=6$
$\sigma=1$	$n=2$	$\theta=0$	0.64	0.63	0.65	0.24	0.93	0.94	0.91	0.83
		$\theta=2$	0.78	0.78	0.86	0.12	0.82	0.82	0.79	0.81
		$\theta=4$	0.86	0.98	0.83	0.15	0.98	0.96	0.96	0.95
		$\theta=6$	1.05	1.29	0.90	0.08	1.17	1.14	1.14	1.01
	$n=3$	$\theta=0$	0.97	0.98	0.67	0.37	1.15	1.15	1.16	1.00
		$\theta=2$	1.13	1.07	0.79	0.38	1.31	1.33	1.29	1.14
		$\theta=4$	1.12	1.07	0.76	0.34	1.32	1.35	1.30	1.25
		$\theta=6$	1.36	1.10	0.68	0.62	1.41	1.39	1.35	1.21
	$n=4$	$\theta=0$	0.43	0.43	0.37	0.18	1.43	1.43	1.36	1.08
		$\theta=2$	0.51	0.51	0.35	0.18	1.79	1.72	1.68	1.17
		$\theta=4$	0.60	0.59	0.37	0.17	2.15	2.04	1.99	0.14
		$\theta=6$	0.75	0.52	0.37	0.17	2.40	2.41	2.36	0.13
	$n=5$	$\theta=0$	0.23	0.35	0.23	0.12	0.51	0.51	0.51	0.52
		$\theta=2$	0.28	0.35	0.23	0.12	0.57	0.59	0.57	0.53
		$\theta=4$	0.31	0.32	0.23	0.12	0.65	0.65	0.64	0.52
		$\theta=6$	0.47	0.35	0.23	0.12	0.83	0.70	0.73	0.72
$\sigma=2$	$n=2$	$\theta=0$	1.08	1.10	0.93	0.17	2.56	2.64	2.82	2.41
		$\theta=2$	1.17	1.10	0.66	0.16	2.87	2.91	3.21	1.67
		$\theta=4$	1.21	1.06	0.34	0.00	3.25	3.39	3.88	1.11
		$\theta=6$	1.13	0.79	0.22	0.00	3.91	4.34	3.88	0.77
	$n=3$	$\theta=0$	1.25	1.26	1.16	0.45	2.63	2.81	2.90	2.27
		$\theta=2$	1.62	1.63	0.96	0.48	3.10	3.03	2.58	2.61
		$\theta=4$	1.65	1.20	0.92	0.47	4.39	4.32	3.34	2.61
		$\theta=6$	1.93	1.10	0.88	0.47	4.33	3.93	3.12	1.01
	$n=4$	$\theta=0$	0.55	0.54	0.50	0.46	2.40	2.40	2.49	1.16
		$\theta=2$	0.61	0.80	0.57	0.43	3.20	3.20	2.80	1.24
		$\theta=4$	1.11	0.93	0.54	0.41	3.51	3.47	3.43	0.90
		$\theta=6$	1.30	0.82	0.54	0.46	3.87	3.40	2.30	0.90
	$n=5$	$\theta=0$	0.45	0.47	0.73	0.37	0.88	0.88	0.59	0.38
		$\theta=2$	0.67	0.53	0.76	0.37	0.87	0.87	0.69	0.38
		$\theta=4$	0.75	0.55	0.76	0.37	1.10	1.03	0.71	0.38
		$\theta=6$	1.10	1.17	0.76	0.37	1.44	1.13	0.71	0.34

Note. $T=15$, $p=10$, $\alpha=0.99$, $D=Normal(3,\sigma)$ *truncated below by 0.*

demand. If outlet 2 stocks a large amount of new inventory, which would make it difficult to sell old inventory, then the old inventory should be rotated to outlet 1. The following theorem is a generalization of Theorem 5.

Theorem 9 *Suppose that $D^2 \geq_{st} D^1$. There is an optimal policy of (3) that satisfies at least one of the following statements:*

(i) $z_1^{1,H} + y^{1,H} \leq y^{2,H}$*;*

(ii) $z_1^{2,H} > 0$ *and either* $z_1^{1,H}$ *or* $y^{2,H}$ *is zero.*

The above theorem offers specific guidelines for the retailer to follow. When there is a lot of old inventory, as much of it as needed at outlet 2 should first be sent there. It is easier to sell the old inventory at outlet 2 because of the greater demand and lower level of new inventory there. In this case, outlet 1 also holds old inventory if and only if the total old inventory exceeds what is needed at outlet 2 and hence $y^{2,H} = 0$ (Part (ii)). When there is only a small amount of old inventory, however, to meet the greater demand at outlet 2, a large amount of new inventory is needed there, which makes it hard to sell the old inventory. In this case, the old inventory should be first sent to outlet 1 (Part (i)). The total inventory at outlet 1 is low not only because the demand is lower there, but also because the level of new inventory there is intentionally kept low to reduce waste.

7 Conclusions

In this study, we explore the idea of transshipment in the context of retailing of perishable goods. Whether or not transshipment is worth implementing depends on its benefits and costs. The practice can increase profit by as much as several percentage points and we identify conditions under which it has the most benefit. Without a rigorous analysis like ours, such level of clarity is impossible. And as long as transshipment can be efficiently integrated into the regular replenishment process, the additional variable costs should be small. Implementation of transshipment may require changes in the business process, IT systems, and employee training upfront. These are all fixed costs, which do not affect our analysis and conclusions. Any retailers interested in the idea can weigh the benefits against the costs upfront and make an informed decision.

When there is a linear transshipment cost, we would expect that the lower the transshipment cost, the more likely separation of inventories may occur. This is confirmed in the numerical studies reported in Table 4. (Here t denotes the unit transshipment cost.) We can also see from the table that when the outdating cost is high or when the ordering cost is high, separation of inventories also occurs. This is expected because under these conditions the need for transshipment to reduce waste is high. Moreover, when the clearance sale price is high, the retailer may clear all the oldest inventory. In this case, we also observe separation of inventories.

Table 4: Separation of inventories in the presence of a positive transshipment cost

		$c = 3$			$c = 5$			$c = 7$		
		$s = 0$	$s = 1$	$s = 2$	$s = 0$	$s = 1$	$s = 2$	$s = 0$	$s = 1$	$s = 2$
$t = 1$	$\theta = 0$	N	N	Y	Y	Y	Y	Y	Y	Y
	$\theta = 1$	Y	Y	Y	Y	Y	Y	Y	Y	Y
	$\theta = 2$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$t = 2$	$\theta = 0$	N	N	Y	N	N	N	Y	Y	Y
	$\theta = 1$	N	N	Y	N	N	N	Y	Y	Y
	$\theta = 2$	N	Y	Y	N	N	N	Y	Y	Y

Note. $T = 10$, $n = 2$, $p = 10$, $\alpha = 0.99$, $D = Uniform[0, 3]$. *"Y" and "N" represent separation and no separation, respectively.*

One possible challenge of implementing transshipment, as we mentioned earlier, stems from the misaligned incentive of store managers. This is not a major problem in convenience store chains (i.e., 7-Eleven Hong Kong) where one owner owns several outlets in close proximity and replenishment is completely centralized. In the extreme case, the incentive issue will go away if the stores are no longer managed by people. Unmanned stores such as Amazon Go in Seattle and BingoBox in Shanghai have generated a lot of discussion (Dastin 2018 and Soo 2017). It would be fascinating to experiment transshipment in those stores.

Waste is a universal problem. Intense discussions have been made and government regulations implemented to increase the cost of waste disposal or even ban food waste in landfills. In our earlier analysis, we assumed that the retailer incurred an outdating cost for every unit of waste. Alternatively, one can also impose a constraint on the expected amount of waste the retailer can generate each period. Regardless, it is a challenge to strike the right balance between the need to reduce waste and the business and consumer interests. Helping retailers to improve their operations, however, may boost their bottom lines and at the same time reduce waste. This is an area where operations researchers can do more.

Appendix

Proof of Lemma 1: Since the state variable x_j appears in the linear term $s\sum_{j=1}^{n-1} x_j$ and in the constraint $z_j^1 + z_j^2 \le x_j$, it is easy to see that $v_t(\mathbf{x}+\delta\mathbf{e}_j) - v_t(\mathbf{x}) \ge s\delta$. This result shows that optimal profit is higher when initial inventory is higher.

(i) The proof is achieved by induction. The result obviously holds for $v_{T+1}(\mathbf{x})$. Suppose $v_{t+1}(\mathbf{x}+\delta\mathbf{e}_i) \le v_{t+1}(\mathbf{x}+\delta\mathbf{e}_{i+1})$ for $1 \le i \le n-2$. We first show that $v_t(\mathbf{x}+\delta\mathbf{e}_1) \le v_t(\mathbf{x}+\delta\mathbf{e}_2)$. Let $(\bar{\mathbf{z}}^1, \bar{q}^1)$ and $(\bar{\mathbf{z}}^2, \bar{q}^2)$ be the optimal solution for the state $\mathbf{x}+\delta\mathbf{e}_1$. If both $\bar{z}_1^1$ and $\bar{z}_1^2$ equal zero, then $(\bar{\mathbf{z}}^1, \bar{q}^1)$ and $(\bar{\mathbf{z}}^2, \bar{q}^2)$ are feasible for the state $\mathbf{x}+\delta\mathbf{e}_2$, and hence obviously $v_t(\mathbf{x}+\delta\mathbf{e}_2) \ge v_t(\mathbf{x}+\delta\mathbf{e}_1)$. Suppose either $\bar{z}_1^1$ or $\bar{z}_1^2$ is positive, and without loss of generality, let us assume $\bar{z}_1^1 > 0$. We can construct a new policy $(\mathbf{z}^1, q^1)$ and $(\mathbf{z}^2, q^2)$ where $(\mathbf{z}^2, q^2) = (\bar{\mathbf{z}}^2, \bar{q}^2)$, $z_1^1 = \bar{z}_1^1 - \delta$, $z_2^1 = \bar{z}_2^1 + \delta$, $z_j^1 = \bar{z}_j^1$ and $q^1 = \bar{q}^1$ for $j \ge 3$. This new policy is feasible for $\mathbf{x}+\delta\mathbf{e}_2$, and

$$
\begin{aligned}
v_t(\mathbf{x}+\delta\mathbf{e}_2) - v_t(\mathbf{x}+\delta\mathbf{e}_1) &\ge \theta\mathsf{E}(\bar{q}^1 + \sum_{j=2}^{n-1} \bar{z}_j^1 + \delta - D^1)^+ - \theta\mathsf{E}(\bar{q}^1 + \sum_{j=2}^{n-1} \bar{z}_j^1 - D^1)^+ + \Delta \\
&\ge 0,
\end{aligned}
$$

where $\Delta = \alpha\mathsf{E}v_{t+1}(\sum_{i=1}^{2} \mathbf{Y}^i(q^i, \mathbf{z}^i, D^i)) - \alpha\mathsf{E}v_{t+1}(\sum_{i=1}^{2} \mathbf{Y}^i(\bar{q}^i, \bar{\mathbf{z}}^i, D^i))$ is the additional profit from future periods from using the new policy. The second inequality holds since the new policy leads to a higher initial inventory with a one-period lifetime in the next period, and hence the additional profit from future periods Δ is positive.

To show that $v_t(\mathbf{x}+\delta\mathbf{e}_i) \le v_t(\mathbf{x}+\delta\mathbf{e}_{i+1})$, let $(\bar{\mathbf{z}}^1, \bar{q}^1)$ and $(\bar{\mathbf{z}}^2, \bar{q}^2)$ be the optimal solution for the state $\mathbf{x}+\delta\mathbf{e}_i$. If both $\bar{z}_i^1$ and $\bar{z}_i^2$ equal zero, then $(\bar{\mathbf{z}}^1, \bar{q}^1)$ and $(\bar{\mathbf{z}}^2, \bar{q}^2)$ are feasible for the state $\mathbf{x}+\delta\mathbf{e}_{i+1}$, and hence obviously $v_t(\mathbf{x}+\delta\mathbf{e}_{i+1}) \ge v_t(\mathbf{x}+\delta\mathbf{e}_i)$. If either $\bar{z}_i^1$ or $\bar{z}_i^2$ is positive, we can similarly construct a new policy $(\mathbf{z}^1, q^1)$ and $(\mathbf{z}^2, q^2)$ where $(\mathbf{z}^2, q^2) = (\bar{\mathbf{z}}^2, \bar{q}^2)$, $z_i^1 = \bar{z}_i^1 - \delta$, $z_{i+1}^1 = \bar{z}_{i+1}^1 + \delta$, $z_j^1 = \bar{z}_j^1$ and $q^1 = \bar{q}^1$ for $j < i$ and $j > i+1$. This new policy is feasible for $\mathbf{x}+\delta\mathbf{e}_{i+1}$, and it leads to a fresher initial inventory in the next period. Hence by the induction hypothesis, we have $v_t(\mathbf{x}+\delta\mathbf{e}_{i+1}) \ge v_t(\mathbf{x}+\delta\mathbf{e}_i)$.

(ii) Let $(\bar{\mathbf{z}}^1, \bar{q}^1)$ and $(\bar{\mathbf{z}}^2, \bar{q}^2)$ be the optimal solution for the state $\mathbf{x}+\delta\mathbf{e}_{n-1}$. If both $\bar{z}_{n-1}^1$ and $\bar{z}_{n-1}^2$ equal zero, then $(\bar{\mathbf{z}}^1, \bar{q}^1)$ and $(\bar{\mathbf{z}}^2, \bar{q}^2)$ are feasible for the state $\mathbf{x}$, and hence

$$
\begin{aligned}
v_t(\mathbf{x}+\delta\mathbf{e}_{n-1}) - v_t(\mathbf{x}) &\le s\delta \\
&\le c\delta.
\end{aligned}
$$

The first inequality holds because $v_t(\mathbf{x})$ is greater than or equal to the profit under any feasible policy. Otherwise, let us assume $\bar{z}_{n-1}^1 > 0$. We can construct a new policy $(\mathbf{z}^1, q^1)$ and $(\mathbf{z}^2, q^2)$

where $(\mathbf{z}^2, q^2) = (\bar{\mathbf{z}}^2, \bar{q}^2)$, $z^1_{n-1} = \bar{z}^1_{n-1} - \delta$, $q^1 = \bar{q}^1 + \delta$, $z^1_j = \bar{z}^1_j$ and $q^1 = \bar{q}^1$ for $j < n-1$. The new policy is feasible for the state $\mathbf{x}$, and

$$\begin{aligned} v_t(\mathbf{x} + \delta \mathbf{e}_{n-1}) - v_t(\mathbf{x}) &\leq s\delta - s\delta + c\delta + \Delta \\ &\leq c\delta, \end{aligned}$$

where $-\Delta = \alpha \mathsf{E} v_{t+1}(\sum_{i=1}^2 \mathbf{Y}^i(q^i, \mathbf{z}^i, D^i)) - \alpha \mathsf{E} v_{t+1}(\sum_{i=1}^2 \mathbf{Y}^i(\bar{q}^i, \bar{\mathbf{z}}^i, D^i))$ is the additional profit from future periods from using the new policy. The second inequality holds since the new policy leads to fresher initial inventories in the next period, and hence $-\Delta$ is non-negative by part (i).

(iii) The first-order derivative of $J_t(\mathbf{z}^i, q^i)$ with respect to z^i_1 is

$$\frac{\partial J_t(\mathbf{z}^i, q^i)}{\partial z^i_1} = (p - s) - (p + \theta)\Phi(q^i + \sum_{j=1}^{n-1} z^i_j).$$

The result follows because the derivative is decreasing in q^i and z^i_j. Furthermore, we can see that $J_t(\mathbf{z}^i, q^i)$ is submodular in $(z^i_1, q^i + \sum_{j=l}^{n-1} z^i_j)$ for any $l \geq 2$. □

Proof of Theorem 1: (i) The proof is achieved by contradiction. Suppose for all optimal policies we have $\bar{q}^1 > 0$ or $\bar{q}^2 > 0$ or $\bar{z}^1_j > 0$ or $\bar{z}^2_j > 0$ for some $j < i$.

If $\bar{q}^1 > 0$, we let $\delta = \min(x_i - \bar{z}^1_i - \bar{z}^2_i, \bar{q}^1) > 0$ and construct a new policy $\mathbf{z}^1 = \bar{\mathbf{z}}^1 + \delta \mathbf{e}_i$, and $q^1 = \bar{q}^1 - \delta$. The new policy will lead to an immediate profit increase of $(c - s)\delta$. However, the new policy will result in less fresh initial inventory in the next period, but the loss is smaller than $\alpha(c - s)\delta$. Hence the new policy will increase the profit, which is a contradiction. Therefore, $\bar{q}^1 = 0$ and similarly we can show that $\bar{q}^2 = 0$.

If $\bar{z}^1_j > 0$ for some $j < i$, we let $\delta = \min(x_i - \bar{z}^1_i - \bar{z}^2_i, \bar{z}^1_j) > 0$ and construct a new policy $\mathbf{z}^1 = \bar{\mathbf{z}}^1 - \delta \mathbf{e}_j + \delta \mathbf{e}_i$, and $q^1 = \bar{q}^1$. The new policy will increase profit by $\theta(\mathsf{E}(\bar{y}^1 + \delta - D^1)^+ - \mathsf{E}(\bar{y}^1 - D^1)^+)$ in the current period and will lead to fresher initial inventory in the next period, which is a contradiction. Thus, $\bar{z}^1_j = 0$ and similarly we can show that $\bar{z}^2_j = 0$.

(ii) Suppose that $\sum_{i=2}^{n-1} x_i \geq 2l_0$. If there exists an $i \geq 2$ such that $\bar{z}^1_i + \bar{z}^2_i < x_i$, then the result holds according to part (i) of this theorem. Otherwise, we have $\bar{z}^1_i + \bar{z}^2_i = x_i$ for all $2 \leq i \leq n-1$. Because $\sum_{i=2}^{n-1} x_i \geq 2l_0$, we have $\sum_{i=2}^{n-1} \bar{z}^1_i + \sum_{i=2}^{n-1} \bar{z}^2_i \geq 2l_0$, which means either $\sum_{i=2}^{n-1} \bar{z}^1_i \geq l_0$ or $\sum_{i=2}^{n-1} \bar{z}^2_i \geq l_0$. The derivative of $J_t(z^i, q^i)$ with respect to z^i_1 is

$$(p + \theta)(\Phi(l_0) - \Phi(z^i_1 + \sum_{j=2}^{n-1} z^i_j + q^i)),$$

which is negative if $\sum_{j=2}^{n-1} z^i_j \geq l_0$. Therefore, when $\sum_{j=2}^{n-1} \bar{z}^i_j \geq l_0$, either $\bar{z}^1_1$ or $\bar{z}^2_1$ is zero. □

Proof of Theorem 2: The proof is by contradiction. For ease of exposition, we prove the result for $n = 3$. The analysis can be easily extended to $n > 3$. Suppose that in all optimal policies, $\bar{z}_1^1 > 0$, $\bar{z}_1^2 > 0$, $\bar{q}^1 > 0$, and $\bar{q}^2 > 0$. Without loss of generality, assume $\bar{q}^1 \geq \bar{q}^2$.

Let $\bar{z}_2 = \bar{z}_2^1 = \bar{z}_2^2$. We construct a new policy:

$$z_1^1 = \bar{z}_1^1 - \delta,\ q^1 = \bar{q}^1 + \delta,\ z_2^1 = \bar{z}_2^1 = \bar{z}_2,$$
$$z_1^2 = \bar{z}_1^2 + \delta,\ q^2 = \bar{q}^2 - \delta,\ z_2^2 = \bar{z}_2^2 = \bar{z}_2.$$

It is not difficult to see that the new policy is feasible as long as $0 \leq \delta \leq \min\{\bar{z}_1^1, \bar{q}^2\}$. Let $\bar{y}^i = \bar{z}_2^i + \bar{q}^i$ for $i = 1, 2$. Then $\bar{y}^1 \geq \bar{y}^2$. The value function under the new policy is given by

$$\begin{aligned}
& \sum_{i=1}^{n-1} s(x_i - \bar{z}_i^1 - \bar{z}_i^2) - c(\bar{q}^1 + \bar{q}^2) \\
+ \quad & p\mathsf{E}\min\{\bar{z}_1^1 + \bar{y}^1, D^1\} + p\mathsf{E}\min\{\bar{z}_2^1 + \bar{y}^2, D^2\} \\
- \quad & [\theta\mathsf{E}(\bar{z}_1^1 + \bar{y}^1 - D^1)^+ + \theta\mathsf{E}(\bar{z}_1^2 + \bar{y}^2 - D^2)^+ - \theta\mathsf{E}(\bar{y}^1 + \delta - D^1)^+ - \theta\mathsf{E}(\bar{y}^2 - \delta - D^2)^+] \\
+ \quad & \alpha f(\delta),
\end{aligned}$$

where

$$f(\delta) = \mathsf{E}v_{t+1}((\bar{y}^1+\delta-D^1)^+ + (\bar{y}^2-\delta-D^2)^+ - (\bar{q}^1+\delta-D^1)^+ - (\bar{q}^2-\delta-D^2)^+, (\bar{q}^1+\delta-D^1)^+ + (\bar{q}^2-\delta-D^2)^+).$$

The new policy leads to the same total ordering cost and total revenue from regular and clearance sales in the current period as those under the optimal policy. The total expected outdating cost under the new policy is decreasing in δ because its derivative with respect to δ is equal to

$$\theta[\Phi(\bar{y}^2 - \delta) - \Phi(\bar{y}^1 + \delta)],$$

which is negative for $\delta \geq 0$.

Let V_i denote the partial derivative $\frac{\partial v_{t+1}}{\partial x_i}$. Let $\bar{\Phi} = 1 - \Phi$. We can show that the first order derivative of $f(\delta)$ is

$$\begin{aligned}
\frac{\partial f(\delta)}{\partial\delta} &= \bar{\Phi}(\bar{y}^2-\delta)\left(\int_0^{\bar{q}^1+\delta} V_2(\bar{z}_2,x_1)\phi(\bar{q}^1+\delta-x_1)dx_1 + \int_0^{\bar{z}_2} V_1(x_1,0)\phi(\bar{y}^1+\delta-x_1)dx_1\right) \\
&\quad -\bar{\Phi}(\bar{y}^1+\delta)\left(\int_0^{\bar{q}^2-\delta} V_2(\bar{z}_2,x_2)\phi(\bar{q}^2-\delta-x_2)dx_2 + \int_0^{\bar{z}_2} V_1(x_2,0)\phi(\bar{y}^2-\delta-x_2)dx_2\right) \\
&\quad + \int_0^{\bar{q}^1+\delta}\int_0^{\bar{z}_2}(V_2-V_1)(\bar{z}_2+x_1,x_2)\phi(\bar{y}^2-\delta-x_1)dx_1\phi(\bar{q}^1+\delta-x_2)dx_2 \\
&\quad - \int_0^{\bar{q}^2-\delta}\int_0^{\bar{z}_2}(V_2-V_1)(\bar{z}_2+x_1,x_2)\phi(\bar{y}^1+\delta-x_1)dx_1\phi(\bar{q}^2-\delta-x_2)dx_2 \\
&= \Delta + \bar{\Phi}(\bar{y}^2-\delta)\bar{\Phi}(\bar{y}^1+\delta)[\int_0^{\bar{q}^2-\delta} V_2(\bar{z}_2,x)(\frac{\phi(\bar{q}^1+\delta-x)}{\bar{\Phi}(\bar{y}^1+\delta)} - \frac{\phi(\bar{q}^2-\delta-x)}{\bar{\Phi}(\bar{y}^2-\delta)})dx \\
&\quad + \int_0^{\bar{z}_2} V_1(x,0)(\frac{\phi(\bar{y}^1+\delta-x)}{\bar{\Phi}(\bar{y}^1+\delta)} - \frac{\phi(\bar{y}^2-\delta-x)}{\bar{\Phi}(\bar{y}^2-\delta)})dx] \\
&\quad + \int_0^{\bar{q}^2-\delta}\int_0^{\bar{z}_2}(V_2-V_1)(\bar{z}_2+x_1,x_2)\tilde{\Delta}dx_1dx_2
\end{aligned}$$

Here

$$\Delta = \bar{\Phi}(\bar{y}^2-\delta)\int_{\bar{q}^2-\delta}^{\bar{q}^1+\delta} V_2(\bar{z}_2,x_1)\phi(\bar{q}^1+\delta-x_1)dx_1 + \int_{\bar{q}^2-\delta}^{\bar{q}^1+\delta}\int_0^{\bar{z}_2}(V_2-V_1)(\bar{z}_2+x_1,x_2)\phi(\bar{y}^2-\delta-x_1)dx_1\phi(\bar{q}^1+\delta-x_2)dx_2$$

and

$$\tilde{\Delta} = \phi(\bar{y}^2-\delta-x_1)\phi(\bar{q}^1+\delta-x_2) - \phi(\bar{y}^1+\delta-x_1)\phi(\bar{q}^2-\delta-x_2).$$

By Lemma 1, we know that $V_2 \geq V_1 \geq s$, therefore $\Delta \geq 0$. Note that for $0 \leq x_1 \leq \bar{z}_2$ and $0 \leq x_2 \leq \bar{q}^2 - \delta$, we always have $\bar{y}^2 - \delta - x_1 \geq \bar{q}^2 - \delta - x_2$. Because the log-concavity of ϕ implies monotone likelihood ratio, that is, $\phi(x-\theta_2)/\phi(x-\theta_1)$ is increasing in x for any $\theta_1 < \theta_2$, we have

$$\begin{aligned}
\frac{\phi(\bar{y}^2-\delta-x_1)}{\phi(\bar{y}^1+\delta-x_1)} &= \frac{\phi(\bar{y}^2-\delta-x_1)}{\phi(\bar{y}^2-\delta-x_1+\bar{q}^1-\bar{q}^2+2\delta)} \\
&\geq \frac{\phi(\bar{q}^2-\delta-x_2)}{\phi(\bar{q}^2-\delta-x_2+\bar{q}^1-\bar{q}^2+2\delta)} \\
&= \frac{\phi(\bar{q}^2-\delta-x_2)}{\phi(\bar{q}^1+\delta-x_2)}.
\end{aligned}$$

Thus, $\tilde{\Delta} \geq 0$. Log-concavity of density function also implies increasing failure rate, so we have

$$\frac{\phi(\bar{q}^1+\delta-x)}{\bar{\Phi}(\bar{q}^1+\delta-x)} \geq \frac{\phi(\bar{q}^2-\delta-x)}{\bar{\Phi}(\bar{q}^2-\delta-x)}$$

Furthermore, log-concavity of $\phi(x)$ implies that $\bar{\Phi}(x)$ is also log-concave and

$$\bar{y}^1+\delta-(\bar{q}^1+\delta-x) = \bar{y}^2-\delta-(\bar{q}^2-\delta-x) = \bar{z}_2+x,$$

we have

$$log\bar{\Phi}(\bar{y}^1+\delta) - log\bar{\Phi}(\bar{q}^1+\delta-x) \leq log\bar{\Phi}(\bar{y}^2-\delta) - log\bar{\Phi}(\bar{q}^2-\delta-x).$$

Thus,

$$\frac{\bar{\Phi}(\bar{q}^1+\delta-x)}{\bar{\Phi}(\bar{y}^1+\delta)} \geq \frac{\bar{\Phi}(\bar{q}^2-\delta-x)}{\bar{\Phi}(\bar{y}^2-\delta)},$$

and we have

$$\frac{\phi(\bar{q}^1+\delta-x)}{\bar{\Phi}(\bar{y}^1+\delta)} \geq \frac{\phi(\bar{q}^2-\delta-x)}{\bar{\Phi}(\bar{y}^2-\delta)}.$$

We can similarly show

$$\frac{\phi(\bar{y}^1+\delta-x)}{\bar{\Phi}(\bar{y}^1+\delta)} \geq \frac{\phi(\bar{y}^2-\delta-x)}{\bar{\Phi}(\bar{y}^2-\delta)}.$$

Therefore, we can conclude that $f(\delta)$ is increasing in δ and so is the value function. This means that there exists an optimal solution where either $z_1^1 = 0$ or $q^2 = 0$, which is a contradiction. □

Proof of Theorem 3: (i) Let $\bar{y}^i = \sum_{j=2}^{n-1} \bar{z}_j^i + \bar{q}^i$ for $i = 1, 2$. Without loss of generality, suppose in all optimal policies, $\bar{z}_1^1 > 0$, $\bar{z}_1^2 > 0$ and $\bar{z}_1^1 + \bar{y}^1 > \bar{z}_1^2 + \bar{y}^2$. Consider a new policy with $z_1^1 = \bar{z}_1^1 - \delta$, $z_1^2 = \bar{z}_1^2 + \delta$, $q^1 = \bar{q}^1$, $q^2 = \bar{q}^2$ and $z_j^k = \bar{z}_j^k$ for $j \geq 2$ and $k = 1, 2$. The new policy is feasible as long as $0 \leq \delta \leq \bar{z}_1^1$.

It is easy to see that the value function under new policy is given by

$$\sum_{i=1}^{n-1} s(x_i - \bar{z}_i^1 - \bar{z}_i^2) - c(\bar{q}^1 + \bar{q}^2) + p\mathsf{E}\min\{\bar{z}_1^1 + \bar{y}^1 - \delta, D^1\} + p\mathsf{E}\min\{\bar{z}_2^1 + \bar{y}^2 + \delta, D^2\}$$

$$-\theta\mathsf{E}(\bar{z}_1^1 - \delta - (D^1 - \bar{y}^1)^+)^+ - \theta\mathsf{E}(\bar{z}_1^2 + \delta - (D^2 - \bar{y}^2)^+)^+ + \alpha\mathsf{E}v_{t+1}(\sum_{i=1}^{2}\mathbf{Y}^i(\bar{q}^i, \bar{\mathbf{z}}^i, D^i)).$$

Its first order derivative with respect to δ is

$$(p+\theta)[\Phi(\bar{z}_1^1 + \bar{y}^1 - \delta) - \Phi(\bar{z}_1^2 + \bar{y}^2 + \delta)],$$

which is greater than zero as long as $\bar{z}_1^1 + \bar{y}^1 - \delta \geq \bar{z}_1^2 + \bar{y}^2 + \delta$. This means that we can find a new policy that is better than the optimal policy and either $z_1^1 = 0$ or $z_1^1 + y^1 = z_1^2 + y^2$, which is a contradiction.

(ii) The proof is similar to that of (i) and hence omitted. □

Proof of Lemma 2: Let d_t^i denote the demand of outlet i in period t. In the last period T, it is easy to see that under the optimal policy, the total inventory in outlet i after clearance, transshipment and ordering is d^i. Thus

$$v_T(\mathbf{x}) = -c(d_T^1 + d_T^2 - x_1 - \sum_{i=2}^{n-1} x_i)^+ + r(d_T^1 + d_T^2) + s(x_1 + \sum_{i=2}^{n-1} x_i - d_T^1 - d_T^2)^+.$$

In period $T-1$, if $x_1 + \sum_{i=2}^{n-1} x_i < d_{T-1}^1 + d_{T-1}^2$, then it is easy to see that (i) holds and

$$v_{T-1}(\mathbf{x}) = -c(d_{T-1}^1 + d_{T-1}^2 - x_1 - \sum_{i=2}^{n-1} x_i) + r(d_{T-1}^1 + d_{T-1}^2) + (r-c)(d_T^1 + d_T^2).$$

If $x_1 + \sum_{i=2}^{n-1} x_i > d_{T-1}^1 + d_{T-1}^2$ and $\sum_{i=2}^{n-1} x_i < d_{T-1}^1 + d_{T-1}^2$, it is easy to see that (ii) holds and

$$v_{T-1}(\mathbf{x}) = r(d_{T-1}^1 + d_{T-1}^2) + s(x_1 + \sum_{i=2}^{n-1} x_i - d_{T-1}^1 - d_{T-1}^2) + (r-c)(d_T^1 + d_T^2).$$

If $x_1 + \sum_{i=2}^{n-1} x_i > d_{T-1}^1 + d_{T-1}^2$ and $\sum_{i=2}^{n-1} x_i \geq d_{T-1}^1 + d_{T-1}^2$, we can similarly verify that (iii) holds and the total inventory that is carried over to the next period is $\sum_{i=2}^{n-1} x_i - d_{T-1}^1 - d_{T-1}^2$, and so

$$v_{T-1}(\mathbf{x}) = r(d_{T-1}^1 + d_{T-1}^2) + sx_1 - c(\sum_{j=T-1}^{T} (d_j^1 + d_j^2) - \sum_{i=2}^{n-1} x_i)^+ + r(d_T^1 + d_T^2) + s(\sum_{i=2}^{n-1} x_i - \sum_{j=T-1}^{T} (d_j^1 + d_j^2))^+.$$

Note that in period $T-1$, the marginal value of x_1 is strictly smaller than that of x_i for $i > 1$ when $\sum_{i=2}^{n-1} x_i < \sum_{j=T-1}^{T} (d_j^1 + d_j^2)$.

In a general period $t < T-1$, we can similarly verify that both (i) and (ii) hold. We shall just prove (iii). Since there are sufficient inventory with a remaining lifetime greater than one period to cover all demand, there is no reason to order and to hold oldest inventory. Since carrying over one additional unit of inventory to next period generates a marginal benefit greater than s, it does not make sense to clear inventory with a remaining lifetime greater than one period. The total inventory with a remaining lifetime of j periods at the beginning of the next period $t+1$ is given by

$$(z_{j+1}^1 - (d_t^1 - \sum_{k=j+2}^{n-1} z_k^1)^+)^+ + (z_{j+1}^2 - (d_t^2 - \sum_{k=j+2}^{n-1} z_k^2)^+)^+.$$

Since the marginal value of newer inventory can be strictly better than that of older inventory, the optimal transshipment should strive to make the inventory composition in the next period as fresh as possible while serving all the demand in the current period. Because $(x-y)^+$ is submodular in (x, y), the outlet with a smaller demand should receive as much fresh inventory as possible while the other outlet holds just enough inventory to cover its demand. □

Proof of Theorem 4: The value function v_1 is upper bounded by the total expected profit when all the future demands are known before making decisions and inventory is non-perishable;

that is

$$
\begin{aligned}
v_1(\mathbf{x}) &\leq \mathsf{E}\left(p\sum_{j=1}^{T}\alpha^{j-1}\sum_{i=1}^{2}D_j^i + s(\sum_{i=1}^{n-1}x_i - \sum_{j=1}^{T}\sum_{i=1}^{2}D_j^i)^+ - \sum_{t=1}^{T}\alpha^{t-1}c(\sum_{i=1}^{2}D_t^i - (\sum_{i=1}^{n-1}x_i - \sum_{j=1}^{t-1}\sum_{i=1}^{2}D_j^i)^+)^+\right) \\
&= \frac{1-\alpha^T}{1-\alpha}2(p-c)\mu + s\mathsf{E}(\sum_{i=1}^{n-1}x_i - \sum_{j=1}^{T}\sum_{i=1}^{2}D_j^i)^+ + c\sum_{t=1}^{T}\alpha^{t-1}\mathsf{E}\min\{(\sum_{i=1}^{n-1}x_i - \sum_{j=1}^{t-1}\sum_{i=1}^{2}D_j^i)^+, \sum_{i=1}^{2}D_t^i\}.
\end{aligned}
$$

The last equality holds because

$$
(\sum_{i=1}^{2}D_t^i - (\sum_{i=1}^{n-1}x_i - \sum_{j=1}^{t-1}\sum_{i=1}^{2}D_j^i)^+)^+ = \sum_{i=1}^{2}D_t^i - \min\{(\sum_{i=1}^{n-1}x_i - \sum_{j=1}^{t-1}\sum_{i=1}^{2}D_j^i)^+, \sum_{i=1}^{2}D_t^i\}.
$$

Since we choose the best v in a consideration set that includes s, to find a lower bound for $v_1^H(\mathbf{x})$, it suffices to find a lower bound for $v_1^H(\mathbf{x})$ assuming that $v = s$. In the last period T, we have

$$
v_T^H(\mathbf{x}) = s\sum_{j=1}^{n-1}x_j + r(x_1, \sum_{i=2}^{n-1}x_i).
$$

Let $\mathbf{x}_t^H$ denote the state trajectory under the approximation. In period $T-1$,

$$
\begin{aligned}
v_{T-1}^H(\mathbf{x}) &= s\sum_{j=1}^{n-1}x_j + \sum_{i=1}^{2}\Big((c-s)z_{[2]}^{i,H} + J(z_1^{i,H}, y^{i,H}) - \alpha s\mathsf{E}(y^{i,H} - D^i)^+\Big) + \alpha\mathsf{E}v_T^H(\mathbf{x}_T^H) \\
&= s\sum_{j=1}^{n-1}x_j + r(x_1, \sum_{i=2}^{n-1}x_i) + \alpha\mathsf{E}r(x_{T,1}^H, \sum_{j=2}^{n-1}x_{T,j}^H).
\end{aligned}
$$

Suppose that for period $t+1$,

$$
v_{t+1}^H(\mathbf{x}) = s\sum_{j=1}^{n-1}x_j + r(x_1, \sum_{i=2}^{n-1}x_i) + \sum_{k=t+1}^{T-1}\alpha^{k-t}\mathsf{E}r(x_{k+1,1}^H, \sum_{j=2}^{n-1}x_{k+1,j}^H).
$$

Then for period t, we have

$$
\begin{aligned}
v_t^H(\mathbf{x}) &= s\sum_{j=1}^{n-1}x_j + \sum_{i=1}^{2}\Big((c-s)z_{[2]}^{i,H} + J(z_1^{i,H}, y^{i,H}) - \alpha s\mathsf{E}(y^{i,H} - D^i)^+\Big) + \alpha\mathsf{E}v_{t+1}^H(\mathbf{x}_{t+1}^H) \\
&= s\sum_{j=1}^{n-1}x_j + r(x_1, \sum_{i=2}^{n-1}x_i) + \alpha\mathsf{E}r(x_{t+1,1}^H, \sum_{j=2}^{n-1}x_{t+1,j}^H) + \sum_{k=t+1}^{T-1}\alpha^{k-t}\mathsf{E}r(x_{k+1,1}^H, \sum_{j=2}^{n-1}x_{k+1,j}^H) \\
&= s\sum_{j=1}^{n-1}x_j + r(x_1, \sum_{i=2}^{n-1}x_i) + \sum_{k=t}^{T-1}\alpha^{k+1-t}\mathsf{E}r(x_{k+1,1}^H, \sum_{j=2}^{n-1}x_{k+1,j}^H).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v_1^H(\mathbf{x}) &= s\sum_{j=1}^{n-1} x_j + r(x_1, \sum_{i=2}^{n-1} x_i) + \sum_{k=1}^{T-1} \alpha^k \mathsf{E} r(x_{k+1,1}^H, \sum_{j=2}^{n-1} x_{k+1,j}^H) \\
&\geq s\sum_{j=1}^{n-1} x_j + r(x_1, \sum_{i=2}^{n-1} x_i) + \frac{\alpha(1-\alpha^{T-1})}{1-\alpha} r(0,0).
\end{aligned}
$$

Let

$$
o(x) = s\mathsf{E}(x - \sum_{j=1}^{T}\sum_{i=1}^{2} D_j^i)^+ + c\sum_{t=1}^{T} \alpha^{t-1} \mathsf{E}\min\{(x - \sum_{j=1}^{t-1}\sum_{i=1}^{2} D_j^i)^+, \sum_{i=1}^{2} D_t^i\}.
$$

Given the lower bound and upper bound, we have

$$
\frac{v_1(\mathbf{x}) - v_1^H(\mathbf{x})}{v_1(\mathbf{x})} \leq 1 - \frac{s\sum_{j=1}^{n-1} x_j + r(x_1, \sum_{i=2}^{n-1} x_i) + \frac{\alpha(1-\alpha^{T-1})}{1-\alpha} r(0,0)}{\frac{1-\alpha^T}{1-\alpha} 2(p-c)\mu + o(\sum_{j=1}^{n-1} x_j)}.
$$

(i) This part follows immediately by letting $\mathbf{x} = 0$ in the inequality above.

(ii) Suppose that $\alpha \to 1$, then the bound approaches

$$
1 - \frac{s\sum_{j=1}^{n-1} x_j + r(x_1, \sum_{i=2}^{n-1} x_i) + (T-1) r(0,0)}{2T(p-c)\mu + \lim_{\alpha\to 1} o(\sum_{j=1}^{n-1} x_j)},
$$

which is smaller than

$$
\begin{aligned}
& 1 - \frac{Tr(0,0)}{2T(p-c)\mu + \lim_{\alpha\to 1} o(\sum_{j=1}^{n-1} x_j)} \\
= \;& \frac{(p-c)\mu - r(0,0)/2}{(p-c)\mu + \lim_{\alpha\to 1} o(\sum_{j=1}^{n-1} x_j)/T}.
\end{aligned}
$$

(iii) Suppose that the demand is compound Poisson, where λ is the arrival rate of customers and each customer purchases a batch size that is independently and identically distributed with mean m and standard deviation σ. Then the mean and variance of the demand D^i are λm and $\lambda(m^2 + \sigma^2)$, respectively. First observe that the bound is smaller than

$$
G(\lambda) = 1 - \frac{\frac{1-\alpha^T}{1-\alpha} r(0,0)}{\frac{1-\alpha^T}{1-\alpha} 2(p-c)\lambda m + s\sum_{i=1}^{n-1} x_i + \frac{1-\alpha^T}{1-\alpha} c \sum_{i=1}^{n-1} x_i}.
$$

We have

$$
\begin{aligned}
r(0,0) &= 2\max_{y\geq 0}\{-cy + p\mathsf{E}\min\{D^i, y\} + s(y - D^i)^+)\} \\
&\geq 2(p-c)\lambda m - 2(p-s)\mathsf{E}(\lambda m - D^i)^+ \\
&\geq 2(p-c)\lambda m - (p-s)\sqrt{\lambda(m^2+\sigma^2)}.
\end{aligned}
$$

The last inequality holds because from Gallego (1992), we know that $\mathsf{E}(\lambda m - D^i)^+ \le \frac{\sqrt{\lambda(m^2+\sigma^2)}}{2}$. Therefore,

$$G(\lambda) \le 1 - \frac{\frac{1-\alpha^T}{1-\alpha}\left(2(p-c)\lambda m - (p-s)\sqrt{\lambda(m^2+\sigma^2)}\right)}{\frac{1-\alpha^T}{1-\alpha}2(p-c)\lambda m + s\sum_{i=1}^{n-1} x_i + \frac{1-\alpha^T}{1-\alpha}c\sum_{i=1}^{n-1} x_i}.$$

As $\lambda \to \infty$, the RHS of the above inequality approaches 0. □

Lemma 3

(i) $J(z_1, y)$ is submodular in (z_1, y);

(ii) If D has a PF_2 distribution, then $J(z_1, y)$ is quasiconcave in y.

Proof of Lemma 3: (i) For any D,

$$(z_1 - (D-y)^+)^+ = (z_1 + y - D)^+ - (y - D)^+,$$

and

$$\min(D, z_1 + y) = z_1 + y - (z_1 + y - D)^+.$$

So

$$J(z_1, y) = (p-s)z_1 + (p-c)y - (p+\theta)\mathsf{E}(z_1 + y - D)^+ + (\theta + v)\mathsf{E}(y - D)^+.$$

The first, second and last terms depend on only one variable, and they are hence submodular. The third is a concave function of $z_1 + y$ and is therefore submodular in (z_1, y) (Lemma 2.6.2, Topkis 1998). The result follows because the sum of submodular functions is still submodular.

(ii) Let

$$f(y) = (p-s)z_1 + (p-c)y + (p-c)\mu - (p+\theta)(z_1+y)^+ + (\theta+v)y^+.$$

Then $J(z_1, y) = \mathsf{E}f(y - D)$. It is easy to show that $f(y)$ is quasiconcave; it is first increasing, then decreasing, and finally decreasing but with a more gentle slope. The result hence follows. □

Throughout the rest of the appendix, to simplify notation, we use x_2, instead of $x_{[2]}$, to represent the total inventory with a life time of two periods or longer. We use $(\bar{z}_1^i, \bar{y}^i)$, $i = 1, 2$, instead of the notations with superscript "H", to denote the decisions under the approximation.
Proof of Theorem 5: The proof is achieved by contradiction. Suppose that in all optimal policies, $\bar{z}_1^1 > 0$, $\bar{z}_1^2 > 0$, $\bar{y}^1 > 0$, and $\bar{y}^2 > 0$. Without loss of generality, assume $\bar{y}^1 \ge \bar{y}^2$.

Let $\delta = \min(\bar{z}_1^1, \bar{y}^2)$ and we construct a new policy:

$$z_1^1 = \bar{z}_1^1 - \delta,\ z_2^1 = \bar{z}_2^1 + \min(\delta, \bar{z}_2^2),\ y^1 = \bar{y}^1 + \delta,$$
$$z_1^2 = \bar{z}_1^2 + \delta,\ z_2^2 = \bar{z}_2^2 - \min(\delta, \bar{z}_2^2),\ y^2 = \bar{y}^2 - \delta.$$

It is not difficult to show that the new policy is still feasible and either z_1^1 or y^2 is zero and $z_1^1, z_2^1, y^1, z_1^2, z_2^2$, and y^2 are all nonnegative. The objective function J can be written as

$$J(z_1^i, y^i) = -sz_1^i - cy^i + p\mathsf{E}\min(D^i, z_1^i + y^i) - \theta\mathsf{E}(z_1^i + y^i - D^i)^+ + (\theta + \alpha v)\mathsf{E}(y^i - D^i)^+.$$

The expected profit under the new policy minus that under the optimal policy is

$$(\theta + \alpha v)[\mathsf{E}(\bar{y}^1 + \delta - D^1)^+ + \mathsf{E}(\bar{y}^2 - \delta - D^2)^+ - \mathsf{E}(\bar{y}^1 - D^1)^+ - \mathsf{E}(\bar{y}^2 - D^2)^+].$$

Because the function $\mathsf{E}(x - D^i)^+$ is a convex function, the above expression is positive. This means that the new policy is also optimal, which is a contradiction.

(i) When $\bar{z}_1^1 + \bar{z}_1^2 = 0$, we solve the following optimization problem:

$$s(x_1 + x_2) + \max_{z_2^1 + z_2^2 \le x_2, z_2^i \ge 0} \{(c - s)(z_2^1 + z_2^2) + J(0, z_2^1) + J(0, z_2^2)\}.$$

The function $(c - s)z + J(0, z)$ is concave in z. Suppose there is an optimal solution such that $\bar{z}_2^1 \ne \bar{z}_2^2$. Then, the symmetric solution $(\frac{\bar{z}_2^1 + \bar{z}_2^2}{2}, \frac{\bar{z}_2^1 + \bar{z}_2^2}{2})$ is also an optimal solution.

(ii) follows from Theorem 3. (iii) is obvious. □

Because of Theorems 1 and 5, the optimization problem (3) is equivalent to the following:

$$\max\{K_1(x_1, x_2), K_2(x_1, x_2), K_3(x_1, x_2)\} \tag{4}$$

where

$$K_1(x_1, x_2) = \max_{z_1^1 + z_1^2 \le x_1, y^1 \ge x_2, z_1^i \ge 0} \{(c - s)x_2 + J(z_1^1, y^1) + J(z_1^2, 0)\},$$

$$K_2(x_1, x_2) = \max_{0 \le z_1^2 \le x_1, y^1 + y^2 \ge x_2, y^i \ge 0} \{(c - s)x_2 + J(0, y^1) + J(z_1^2, y^2)\},$$

and

$$K_3(x_1, x_2) = \max_{z_2^1 + z_2^2 \le x_2, z_2^i \ge 0} \{(c - s)(z_2^1 + z_2^2) + J(0, z_2^1) + J(0, z_2^2)\}.$$

Here K_1 represents the case when there are no clearance sales of new items, and all new items are allocated to outlet 1. K_2 represents the case when there are no clearance sales of new items, and all old items are allocated to outlet 2. K_3 represents the case when some or all new items are sold in clearance sales, in which case, no order is placed and all old items are sold in clearance sales. In light of Theorems 1 and 5, these events are collectively exhaustive.

Throughout the rest of the appendix, we use the notations $\bar{z}_k^j(x_1, x_2|K_i)$ and $\bar{y}^j(x_1, x_2|K_i)$ to denote the optimal solutions to the optimization problem K_i in (4). Let $u_0 = \arg\max_{y\geq 0}\{(c-s)y + J(0,y)\}$. The following lemma characterizes the monotonicity of the optimal solution to each of the three maximization problems in (4).

Lemma 4

(i) The function $\bar{z}_1^1(x_1, x_2|K_1)$ is increasing in x_1 and decreasing in x_2, and $\bar{y}^1(x_1, x_2|K_1)$ is decreasing in x_1 and increasing in x_2;

(ii) The function $\bar{z}_1^2(x_1, x_2|K_2)$ is increasing in x_1 and decreasing in x_2, and $\bar{y}^2(x_1, x_2|K_2)$ is decreasing in x_1 and increasing in x_2.

(iii) If $x_2 < 2u_0$, $K_3(x_1, x_2) \leq K_2(x_1, x_2)$.

(iv) If $x_2 \geq 2u_0$, then $\bar{y}^1 = \bar{y}^2 = \bar{z}_2^1 = \bar{z}_2^2 = u_0$, and $\bar{z}_1^1 = \bar{z}_1^2 = 0$.

Proof of Lemma 4: (i) We first look at the optimization problem K_1. Let $\tilde{y}^1 = -y^1$, $\tilde{x}_2 = -x_2$ and $x_1 - z_1^2 = \tilde{z}_1^2$. Then the objective function is supermodular in $(z_1^1, \tilde{z}_1^2, \tilde{y}^1, x_1, \tilde{x}_2)$ and the constraint set forms a lattice. Therefore, $\bar{z}_1^1(x_1, x_2|K_1)$ is increasing in x_1 and decreasing in x_2, and $\bar{y}^1(x_1, x_2|K_1)$ is decreasing in x_1 and increasing in x_2.

(ii) We next look at the optimization problem K_2. Let $\tilde{y}^1 = x_2 - y^1$, $\tilde{x}_1 = -x_1$ and $\tilde{z}_1^2 = -z_1^2$. Then the objective function is supermodular in $(\tilde{z}_1^2, \tilde{y}^1, y_2, \tilde{x}_1, x_2)$ and the constraint set forms a lattice. Therefore, $\bar{y}^2(x_1, x_2|K_2)$ is decreasing in x_1 and increasing in x_2, and $\bar{z}_1^2(x_1, x_2|K_2)$ is increasing in x_1 and decreasing in x_2.

(iii) Note first that $u_0 = \arg\max_{z\geq 0}\{(c-s)z + J(0,z)\}$. When $x_2 < 2u_0$, the optimal solution to K_3 must be a boundary solution; that is, $\bar{z}_2^1(x_1, x_2|K_3) + \bar{z}_2^1(x_1, x_2|K_3) = x_2$. It is easy to see that the optimal solution to K_3 is a feasible but not necessarily optimal solution to K_2.

(iv) For $x_2 \geq 2u_0$, if we can show that $K_1 \leq K_2 \leq K_3$, then the result follows. We can simplify the optimization problem K_1 by sequentially optimizing the objective function with respect to each decision variable, first z_1^2, then z_1^1, and finally y^1. Note that

$$\begin{aligned} \frac{\partial J(z,y)}{\partial z} &= p - s - (p+\theta)\Phi(z+y) \\ &= (p+\theta)(\Phi(l_0) - \Phi(z+y)). \end{aligned}$$

We obtain the following two cases after optimizing over z_1^2.

Case 1: If $x_1 \le l_0$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max_{0 \le z_1^2 \le x_1 - z_1^1, y^1 \ge x_2, z_1^1 \ge 0} \{J(z_1^1, y^1) + J(z_1^2, 0)\} \\ &= \max_{y^1 \ge x_2, 0 \le z_1^1 \le x_1} \{J(z_1^1, y^1) + J(x_1 - z_1^1, 0)\}\}. \end{aligned}$$

Based on the following derivative,

$$\frac{\partial\{J(z_1^1, y^1) + J(x_1 - z_1^1, 0)\}}{\partial z_1^1} = (p+\theta)(\Phi(x_1 - z_1^1) - \Phi(z_1^1 + y^1)),$$

we know that $\arg\max_{z_1^1 \ge 0} J(z_1^1, y^1) + J(x_1 - z_1^1, 0) = (x_1 - y^1)^+/2$. There are two sub-cases after optimizing over z_1^1.

Case 1(i): If $x_2 \le x_1$,

$$K_1 - (c-s)x_2 = \max\{\max_{y^1 \ge x_1} \{J(0, y^1) + J(x_1, 0)\}, \max_{x_2 \le y^1 \le x_1} \{J(\frac{x_1 - y^1}{2}, y^1) + J(\frac{x_1 + y^1}{2}, 0)\}\}.$$

Case 1(ii) If $x_2 > x_1$,

$$K_1 - (c-s)x_2 = \max_{y^1 \ge x_2} \{J(0, y^1) + J(x_1, 0)\}.$$

Case 2: If $x_1 > l_0$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max_{0 \le z_1^2 \le x_1 - z_1^1, y^1 \ge x_2, z_1^1 \ge 0} \{J(z_1^1, y^1) + J(z_1^2, 0)\} \\ &= \max\{\max_{x_1 - z_1^1 \ge l_0, y^1 \ge x_2, z_1^1 \ge 0} \{J(z_1^1, y^1) + J(l_0, 0)\}, \\ &\qquad \max_{x_1 - z_1^1 < l_0, y^1 \ge x_2, z_1^1 \ge 0} \{J(z_1^1, y^1) + J(x_1 - z_1^1, 0)\}\}. \end{aligned}$$

Optimizing over z_1^1, we have the following three sub-cases.

Case 2(i): If $x_2 \le 2l_0 - x_1$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{\max_{2l_0 - x_1 \le y^1 \le l_0} \{J(l_0 - y^1, y^1) + J(l_0, 0)\}, \max_{x_2 \le y^1 \le 2l_0 - x_1} \{J(x_1 - l_0, y^1) + J(l_0, 0)\}, \\ &\qquad \max_{y^1 \ge 2l_0 - x_1} \{J(x_1 - l_0, y^1) + J(l_0, 0)\}, \max_{x_2 \le y^1 \le 2l_0 - x_1} \{J(\frac{x_1 - y^1}{2}, y^1) + J(\frac{x_1 + y^1}{2}, 0)\}\} \\ &= \max\{J(0, l_0) + J(l_0, 0), \max_{y^1 \ge x_2} \{J(x_1 - l_0, y^1) + J(l_0, 0)\}, \\ &\qquad \max_{x_2 \le y^1 \le 2l_0 - x_1} \{J(\frac{x_1 - y^1}{2}, y^1) + J(\frac{x_1 + y^1}{2}, 0)\}\}. \end{aligned}$$

The second equality holds since $J(l_0 - y^1, y^1)$ is convex in y^1 and its derivative is

$$\frac{\partial\{J(l_0 - y^1, y^1)\}}{\partial y^1} = s - c + (\theta + \alpha v)\Phi(y^1).$$

Case 2(ii): If $2l_0 - x_1 \leq x_2 \leq l_0$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{\max_{x_2 \leq y^1 \leq l_0}\{J(l_0 - y^1, y^1) + J(l_0, 0)\}, \max_{y^1 \geq l_0}\{J(0, y^1) + J(l_0, 0)\}, \\ &\quad \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0, 0)\}\}. \end{aligned}$$

Case 2(iii): If $x_2 \geq l_0$,

$$K_1 - (c-s)x_2 = \max\{\max_{y^1 \geq x_2}\{J(0, y^1) + J(l_0, 0)\}, \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0, 0)\}\}.$$

From the above discussion, we know that for $x_2 \geq 2u_0 \geq 2l_0$, either we have Case 2(iii) where

$$K_1 - (c-s)x_2 = \max\{\max_{y^1 \geq x_2}\{J(0, y^1) + J(l_0, 0)\}, \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0, 0)\}\},$$

or we have Case 1(ii) where

$$K_1 - (c-s)x_2 = \max_{y^1 \geq x_2}\{J(0, y^1) + J(x_1, 0)\}.$$

Since $J(0, y^1) \leq J(x_1 - l_0, y^1)$ for any $y^1 \geq x_2 \geq 2u_0$, we have $\bar{z}_1^1(x_1, x_2|K_1) = 0$ and $K_1 \leq K_2$.

We can similarly simplify the optimization problem K_2 by sequentially optimizing the objective function with respect to each decision variable, first y^1, then z_1^2, and finally y^2. Let $u = \Phi^{-1}(\frac{p-c}{p-\alpha v})$. Note that

$$\begin{aligned} \frac{\partial J(0, y)}{\partial y} &= p - c - (p - \alpha v)\Phi(y) \\ &= (p - \alpha v)(\Phi(u) - \Phi(y)). \end{aligned}$$

There are two cases after optimizing over y^1.

Case 1: If $x_2 \leq u$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max_{0 \leq z_1^2 \leq x_1, y^1 + y^2 \geq x_2, y^i \geq 0}\{J(0, y^1) + J(z_1^2, y^2)\} \\ &= \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0}\{J(0, u) + J(z_1^2, y^2)\}. \end{aligned}$$

Optimizing over z_1^2, we further obtain two sub-cases.

Case 1(i): If $x_1 \geq l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0}\{J(0, u) + J(z_1^2, y^2)\} \\ &= \max\{\max_{0 \leq y^2 \leq l_0}\{J(0, u) + J(l_0 - y^2, y^2)\}, \max_{y^2 \geq l_0}\{J(0, u) + J(0, y^2)\}\}. \end{aligned}$$

Case 1(ii): If $x_1 \le l_0$,

$$\begin{aligned} K_2-(c-s)x_2 &= \max_{0\le z_1^2\le x_1, y^2\ge 0}\{J(0,u)+J(z_1^2,y^2)\} \\ &= \max\{\max_{l_0-x_1\le y^2\le l_0}\{J(0,u)+J(l_0-y^2,y^2)\}, \max_{0\le y^2\le l_0-x_1}\{J(0,u)+J(x_1,y^2)\} \\ &\quad \max_{y^2\ge l_0}\{J(0,u)+J(0,y^2)\}\}. \end{aligned}$$

Case 2: If $x_2 \ge u$,

$$\begin{aligned} K_2-(c-s)x_2 &= \max_{0\le z_1^2\le x_1, y^1+y^2\ge x_2, y^i\ge 0}\{J(0,y^1)+J(z_1^2,y^2)\} \\ &= \max\{\max_{0\le z_1^2\le x_1, 0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(z_1^2,y^2)\}, \\ &\quad \max_{0\le z_1^2\le x_1, y^2\ge x_2-u}\{J(0,u)+J(z_1^2,y^2)\}\}. \end{aligned}$$

Optimizing over z_1^2, we obtain the following five sub-cases.

Case 2(i): If $x_1 \ge l_0$ and $x_2-u \ge l_0$,

$$\begin{aligned} K_2-(c-s)x_2 &= \max\{\max_{0\le z_1^2\le x_1, 0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(z_1^2,y^2)\}, \max_{0\le z_1^2\le x_1, y^2\ge x_2-u}\{J(0,u)+J(z_1^2,y^2)\}\} \\ &= \max\{\max_{0\le y^2\le l_0}\{J(0,x_2-y^2)+J(l_0-y^2,y^2)\}, \max_{l_0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(0,y^2)\}, \\ &\quad \max_{y^2\ge x_2-u}\{J(0,u)+J(0,y^2)\}\}. \end{aligned}$$

Case 2(ii): If $x_1 \ge l_0$ and $x_2-u \le l_0$

$$\begin{aligned} K_2-(c-s)x_2 &= \max\{\max_{0\le z_1^2\le x_1, 0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(z_1^2,y^2)\}, \max_{0\le z_1^2\le x_1, y^2\ge x_2-u}\{J(0,u)+J(z_1^2,y^2)\}\} \\ &= \max\{\max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(l_0-y^2,y^2)\}, \max_{x_2-u\le y^2\le l_0}\{J(0,u)+J(l_0-y^2,y^2)\}, \\ &\quad \max_{y^2\ge l_0}\{J(0,u)+J(0,y^2)\}\} \end{aligned}$$

Case 2(iii): If $x_1 \le l_0$ and $l_0-x_1 \ge x_2-u$

$$\begin{aligned} K_2-(c-s)x_2 &= \max\{\max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(x_1,y^2)\}, \max_{l_0-x_1\le y^2\le l_0}\{J(0,u)+J(l_0-y^2,y^2)\}, \\ &\quad \max_{x_2-u\le y^2\le l_0-x_1}\{J(0,u)+J(x_1,y^2)\}, \max_{y^2\ge l_0}\{J(0,u)+J(0,y^2)\}\} \\ &= \max\{\max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(x_1,y^2)\}, \max_{x_2-u\le y^2\le l_0-x_1}\{J(0,u)+J(x_1,y^2)\}, \\ &\quad J(0,u)+J(0,\max\{u,l_0\})\} \end{aligned}$$

Case 2(iv): If $x_1 \le l_0$ and $l_0 - x_1 \le x_2 - u \le l_0$

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{l_0 - x_1 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{x_2 - u \le y^2 \le l_0}\{J(0, u) + J(l_0 - y^2, y^2)\}, \max_{y^2 \ge l_0}\{J(0, u) + J(0, y^2)\}\} \\ &= \max\{\max_{l_0 - x_1 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad J(0, u) + J(0, \max\{u, l_0\})\}. \end{aligned}$$

Case 2(v): If $x_1 \le l_0$ and $x_2 - u \ge l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{l_0 - x_1 \le y^2 \le l_0}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{l_0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(0, y^2)\}, \max_{y^2 \ge x_2 - u}\{J(0, u) + J(0, y^2)\}\}. \end{aligned}$$

From the above discussion, we know that for $x_2 \ge 2u_0$, either we have Case 2(i) where

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{0 \le y^2 \le l_0}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{l_0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(0, y^2)\}, \\ &\quad \max_{y^2 \ge x_2 - u}\{J(0, u) + J(0, y^2)\}\}. \end{aligned}$$

or we have Case 2(v) where

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{l_0 - x_1 \le y^2 \le l_0}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{l_0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(0, y^2)\}, \max_{y^2 \ge x_2 - u}\{J(0, u) + J(0, y^2)\}\}. \end{aligned}$$

Because for $0 \le y^2 \le l_0 - x_1$,

$$\begin{aligned} \frac{\partial\{J(0, x_2 - y^2) + J(x_1, y^2)\}}{\partial y^2} &= (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2) \\ &\ge (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) \\ &\ge (p - \alpha v)\Phi(x_2 + x_1 - l_0) - (p + \theta)\Phi(l_0) \\ &\ge 0 \end{aligned}$$

and for $0 \le y^2 \le l_0$,

$$\begin{aligned} \frac{\partial\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}}{\partial y^2} &= -(p - s) + (p - \alpha v)\Phi(x_2 - y^2) + (\theta + \alpha v)\Phi(y^2) \\ &\ge -(p - s) + (p - \alpha v)\Phi(x_2 - l_0) \\ &\ge 0, \end{aligned}$$

we can simplify K_2 as

$$\begin{aligned} K_2 - (c-s)x_2 &= \max_{l_0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(0, y^2)\} \\ &= J(0, x_2/2) + J(0, x_2/2). \end{aligned}$$

Thus, we have $K_2 \le K_3$. □

We introduce the following result which will be used in later proofs.

Lemma 5 *If $u \le l_0$, then $J(l_0, 0) \ge J(0, u)$.*

Proof of Lemma 5: The inequality $u \le l_0$ is equivalent to $\frac{c-s}{\theta+\alpha v} \ge \Phi(l_0)$. Because $\frac{\partial J(x,0)}{\partial x} \ge \frac{\partial J(0,x)}{\partial x}$ for $\Phi(x) \le \frac{c-s}{\theta+\alpha v}$, we have $J(l_0, 0) \ge J(0, u)$. □

Proof of Theorems 6, 7, and 8: Since Theorems 6, 7, and 8 focus on different aspects of the optimal policy, here we provide a unified proof by first characterizing the optimal policy under different states and then defining appropriate switching curves $B(x_1)$ and $C(x_1)$.

Let $O(x_1) = \arg\max_{y^1 \ge 0} J(x_1, y^1)$ and $l = \Phi^{-1}(\frac{p-c}{p+\theta})$. It is easy to show that $O(x_1)$ is decreasing and $O(x_1) = 0$ for $x_1 \ge l$. Moreover, $O(x_1) \le u$ where $u = \Phi^{-1}(\frac{p-c}{p-\alpha v})$. Based on the values of the parameters, the proof below is divided into three parts. In Part I, $u \le l_0$. In Part II, $u > l_0$ and $J(l_0, 0) \le J(0, u)$. In Part III, $u > l_0$ and $J(l_0, 0) > J(0, u)$.

Part I: From the proof of Lemma 4, we can characterize the optimal policy for the optimization problem K_1 as follows.

Case 1(i): If $x_1 \le l_0$ and $x_2 \le x_1$,

$$K_1 - (c-s)x_2 \quad = \quad \max\{\max_{y^1 \ge x_1}\{J(0, y^1) + J(x_1, 0)\}, \max_{x_2 \le y^1 \le x_1}\{J(\frac{x_1 - y^1}{2}, y^1) + J(\frac{x_1 + y^1}{2}, 0)\}\}.$$

In this case, when $x_1 \le u$, $K_1(x_1, x_2) = (c-s)x_2 + J(0, u) + J(x_1, 0)$; and when $x_1 \ge u$, K_1 can be simplified as:

$$K_1(x_1, x_2) = (c-s)x_2 + \max_{x_2 \le y^1 < x_1}\{J(\frac{x_1 - y^1}{2}, y^1) + J(\frac{x_1 + y^1}{2}, 0)\}.$$

Note that here the optimal $y^1 < x_1$.

Case 1(ii): If $x_1 \le l_0$ and $x_2 > x_1$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max_{y^1 \ge x_2}\{J(0, y^1) + J(x_1, 0)\} \\ &= J(0, \max\{x_2, u\}) + J(x_1, 0). \end{aligned}$$

Case 2(i): If $x_1 \ge l_0$ and $x_2 \le 2l_0 - x_1$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{J(0, l_0) + J(l_0, 0), \max_{y^1 \ge x_2}\{J(x_1 - l_0, y^1) + J(l_0, 0)\}, \\ &\quad \max_{x_2 \le y^1 \le 2l_0 - x_1}\{J(\frac{x_1 - y^1}{2}, y^1) + J(\frac{x_1 + y^1}{2}, 0)\}\} \\ &= J(x_1 - l_0, \max\{x_2, O(x_1 - l_0)\}) + J(l_0, 0) \end{aligned}$$

Case 2(ii): If $x_1 \geq l_0$ and $2l_0 - x_1 \leq x_2 \leq l_0$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{\max_{x_2 \leq y^1 \leq l_0}\{J(l_0 - y^1, y^1) + J(l_0, 0)\}, \max_{y^1 \geq l_0}\{J(0, y^1) + J(l_0, 0)\}, \\ &\quad \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0, 0)\}\} \\ &= J(l_0 - x_2, x_2) + J(l_0, 0). \end{aligned}$$

Case 2(iii): If $x_1 \geq l_0$ and $x_2 \geq l_0$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{\max_{y^1 \geq x_2}\{J(0, y^1) + J(l_0, 0)\}, \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0, 0)\}\} \\ &= J(0, x_2) + J(l_0, 0). \end{aligned}$$

Similarly, we can characterize the optimal policy for the optimization problem K_2.

Case 1(i): If $x_2 \leq u$ and $x_1 \geq l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0}\{J(0, u) + J(z_1^2, y^2)\} \\ &= \max\{\max_{0 \leq y^2 \leq l_0}\{J(0, u) + J(l_0 - y^2, y^2)\}, \max_{y^2 \geq l_0}\{J(0, u) + J(0, y^2)\}\} \\ &= J(0, u) + J(l_0, 0). \end{aligned}$$

Case 1(ii): If $x_2 \leq u$ and $x_1 \leq l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0}\{J(0, u) + J(z_1^2, y^2)\} \\ &= \max\{\max_{l_0 - x_1 \leq y^2 \leq l_0}\{J(0, u) + J(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1}\{J(0, u) + J(x_1, y^2)\} \\ &\quad \max_{y^2 \geq l_0}\{J(0, u) + J(0, y^2)\}\} \\ &= J(0, u) + J(x_1, O(x_1)). \end{aligned}$$

Case 2(i): If $x_1 \geq l_0$ and $x_2 - u \geq l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u}\{J(0, x_2 - y^2) + J(z_1^2, y^2)\}, \max_{0 \leq z_1^2 \leq x_1, y^2 \geq x_2 - u}\{J(0, u) + J(z_1^2, y^2)\}\} \\ &= \max\{\max_{0 \leq y^2 \leq l_0}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{l_0 \leq y^2 \leq x_2 - u}\{J(0, x_2 - y^2) + J(0, y^2)\}, \\ &\quad \max_{y^2 \geq x_2 - u}\{J(0, u) + J(0, y^2)\}\} \\ &= J(0, x_2) + J(l_0, 0). \end{aligned}$$

Case 2(ii): If $x_1 \geq l_0$ and $u \leq x_2 \leq u + l_0$

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u}\{J(0, x_2 - y^2) + J(z_1^2, y^2)\}, \max_{0 \leq z_1^2 \leq x_1, y^2 \geq x_2 - u}\{J(0, u) + J(z_1^2, y^2)\}\} \\ &= \max\{\max_{0 \leq y^2 \leq x_2 - u}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{x_2 - u \leq y^2 \leq l_0}\{J(0, u) + J(l_0 - y^2, y^2)\}, \\ &\quad \max_{y^2 \geq l_0}\{J(0, u) + J(0, y^2)\}\} \\ &= J(0, x_2) + J(l_0, 0). \end{aligned}$$

Case 2(iii): If $x_1 \le l_0$ and $l_0 - x_1 \ge x_2 - u \ge 0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \max_{l_0 - x_1 \le y^2 \le l_0}\{J(0,u) + J(l_0 - y^2, y^2)\}, \\ &\quad \max_{x_2 - u \le y^2 \le l_0 - x_1}\{J(0,u) + J(x_1, y^2)\}, \max_{y^2 \ge l_0}\{J(0,u) + J(0, y^2)\} \\ &= \max\{\max_{0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \max_{l_0 - x_1 \le y^2 \le l_0}\{J(0,u) + J(l_0 - y^2, y^2)\}, \\ &\quad \max_{x_2 - u \le y^2 \le l_0 - x_1}\{J(0,u) + J(x_1, y^2)\}\} \end{aligned}$$

When $u \le l_0$, we can show that $O(x_1) \le l_0 - x_1$. Thus if $x_2 - u \le O(x_1)$, $K_2(x_1, x_2) = (c-s)x_2 + J(0,u) + J(x_1, O(x_1))$.

Case 2(iv): If $x_1 \le l_0$ and $l_0 - x_1 \le x_2 - u \le l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{l_0 - x_1 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{x_2 - u \le y^2 \le l_0}\{J(0,u) + J(l_0 - y^2, y^2)\}, \max_{y^2 \ge l_0}\{J(0,u) + J(0, y^2)\} \\ &= \max\{\max_{l_0 - x_1 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{x_2 - u \le y^2 \le l_0}\{J(0,u) + J(l_0 - y^2, y^2)\}\}. \end{aligned}$$

Case 2(v): If $x_1 \le l_0$ and $x_2 - u \ge l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{\max_{l_0 - x_1 \le y^2 \le l_0}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{l_0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(0, y^2)\}, \max_{y^2 \ge x_2 - u}\{J(0,u) + J(0, y^2)\}\} \\ &= \max\{\max_{l_0 - x_1 \le y^2 \le l_0}\{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1}\{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{l_0 \le y^2 \le x_2 - u}\{J(0, x_2 - y^2) + J(0, y^2)\}\}. \end{aligned}$$

For Cases 2(iii),(iv),(v), we know that if $x_2 - u \ge O(x_1)$, for $0 \le y^2 \le x_2 - u$,

$$\begin{aligned} \frac{\partial\{J(0, x_2 - y^2) + J(x_1, y^2)\}}{\partial y^2} &= (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2) \\ &\ge (p - c) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2), \end{aligned}$$

which is greater than zero for $y^2 \le O(x_1)$, and hence $\bar{y}^2(x_1, x_2 | K_2) > 0$ if $x_1 \le l$. If $x_1 \ge u$ and $x_1 \ge x_2$,

$$\begin{aligned} \frac{\partial\{J(0, x_2 - y^2) + J(x_1, y^2)\}}{\partial y^2} &= (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2) \\ &\le (p - \alpha v)\Phi(x_2) - (p + \theta)\Phi(x_1) + (\theta + \alpha v)\Phi(x_2 - u) \\ &\le (\theta + \alpha v)(\Phi(x_2 - u) - \Phi(x_1)) \\ &\le 0, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial\{J(0,x_2-y^2)+J(l_0-y^2,y^2)\}}{\partial y^2} &= -(p-s)+(p-\alpha v)\Phi(x_2-y^2)+(\theta+\alpha v)\Phi(y^2)\\ &\le -(p-s)+(p-\alpha v)\Phi(x_2+x_1-l_0)+(\theta+\alpha v)\Phi(x_2-u)\\ &\le -(p-s)+(p+\theta)\Phi(x_2)\\ &\le 0,\end{aligned}$$

and hence $\bar{y}^2(x_1,x_2|K_2)=0$.

From the above discussion, if we let

$$C(x_1) = \begin{cases} 0 & \text{if} \quad x_1 \le u \\ x_1 & \text{if} \quad u \le x_1 \le l_0 \\ l_0 & \text{if} \quad x_1 \ge l_0 \end{cases}$$

and $B(x_1)=\sup\{x_2\ge 0 : \bar{y}^2(x_1,x_2|K_2)=0\}$. Then, all of the results in the theorems hold.

Part II: For the optimization problem K_1, it is easy to see that if $x_1\ge l_0$ and $x_2\le u$, then $K_1(x_1,x_2)=(c-s)x_2+J(0,u)+J(l_0,0)$. Hence $\bar{z}_1^1(x_1,x_2|K_1)=0$ for any x_1 and x_2.

For the optimization K_2, we consider the following cases.

Case 1. If $x_2\le 2u$, then $K_2(x_1,x_2)=(c-s)x_2+J(0,u)+J(0,u)$. Since $\bar{z}_1^2(x_1,x_2|K_2)$ is decreasing in x_2, $\bar{z}_1^2(x_1,x_2|K_2)=0$ for all x_2.

Case 2. If $2u\le x_2\le 2u_0$, then $K_2(x_1,x_2)=(c-s)x_2+J(0,x_2/2)+J(0,x_2/2)$.

From the above discussion, if we define $B(x_1)=C(x_1)=0$, then all of the results in the theorem hold.

Part III: Since $J(l_0-y,y)$ is convex in y, we let $\tilde{l}=\sup\{y\in[0,l_0] : J(l_0-y,y)\ge J(0,u)\}$. It is easy to see that $s-c+(\theta+\alpha v)\Phi(\tilde{l})\le 0$. We first derive the optimal policy for the optimization problem K_1.

Case 1(i): If $x_1\le l_0$ and $x_2\le x_1$,

$$K_1-(c-s)x_2 = \max\{\max_{y^1\ge x_1}\{J(0,y^1)+J(x_1,0)\}, \max_{x_2\le y^1\le x_1}\{J(\frac{x_1-y^1}{2},y^1)+J(\frac{x_1+y^1}{2},0)\}\}.$$

In this case, if $x_1\le l$, $K_1(x_1,x_2)=(c-s)x_2+J(0,u)+J(x_1,0)$. If $x_1\ge l$,

$$K_1(x_1,x_2) = \max\{J(0,u)+J(x_1,0), \max_{x_2\le y^1\le x_1}\{J(\frac{x_1-y^1}{2},y^1)+J(\frac{x_1+y^1}{2},0)\}\}.$$

Case 1(ii): If $x_1\le l_0$ and $x_2>x_1$,

$$\begin{aligned}K_1-(c-s)x_2 &= \max_{y^1\ge x_2}\{J(0,y^1)+J(x_1,0)\}\\ &= J(0,\max\{x_2,u\})+J(x_1,0).\end{aligned}$$

Case 2(i): If $x_1 \geq l_0$ and $x_2 \leq 2l_0 - x_1$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{J(0,l_0) + J(l_0,0), \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0,0)\}, \\ &\quad \max_{x_2 \leq y^1 \leq 2l_0 - x_1}\{J(\frac{x_1 - y^1}{2}, y^1) + J(\frac{x_1 + y^1}{2}, 0)\}\}. \end{aligned}$$

Case 2(ii): If $x_1 \geq l_0$ and $2l_0 - x_1 \leq x_2 \leq l_0$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{\max_{x_2 \leq y^1 \leq l_0}\{J(l_0 - y^1, y^1) + J(l_0,0)\}, \max_{y^1 \geq l_0}\{J(0,y^1) + J(l_0,0)\}, \\ &\quad \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0,0)\}\} \\ &= J(l_0,0) + \max\{J(0,u), J(l_0 - x_2, x_2)\}. \end{aligned}$$

Thus, if $x_2 \geq \tilde{l}$, $K_1 = (c-s)x_2 + J(0,u) + J(l_0,0)$. If $x_2 \leq \tilde{l}$, $K_1 = (c-s)x_2 + J(l_0 - x_2, x_2) + J(l_0,0)$.

Case 2(iii): If $x_1 \geq l_0$ and $x_2 \geq l_0$,

$$\begin{aligned} K_1 - (c-s)x_2 &= \max\{\max_{y^1 \geq x_2}\{J(0,y^1) + J(l_0,0)\}, \max_{y^1 \geq x_2}\{J(x_1 - l_0, y^1) + J(l_0,0)\}\} \\ &= J(0, \max\{u, x_2\}) + J(l_0,0). \end{aligned}$$

Similarly, we can characterize the optimal policy for the optimization problem K_2. Notice that

$$\frac{\partial J(x_1,y)}{\partial y}|_{y=l_0-x_1} = (\theta + \alpha v)(\Phi(l_0 - x_1) - \frac{c-s}{\theta + \alpha v}).$$

Let $\tilde{u} = \sup\{x_1 : \Phi(l_0 - x_1) \geq \frac{c-s}{\theta+\alpha v}\}$. Thus, $\max_{0 \leq y^2 \leq l_0 - x_1} J(x_1, y^2) = J(x_1, l_0 - x_1)$ for $x_1 \leq \tilde{u}$ and $\max_{0 \leq y^2 \leq l_0 - x_1} J(x_1, y^2) = J(x_1, O(x_1))$ otherwise. Let $u' = \inf\{x_1 \geq \tilde{u} : J(x_1, O(x_1)) \geq J(0,u)\}$.

Case 1(i): If $x_2 \leq u$ and $x_1 \geq l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0}\{J(0,u) + J(z_1^2, y^2)\} \\ &= \max\{\max_{0 \leq y^2 \leq l_0}\{J(0,u) + J(l_0 - y^2, y^2)\}, \max_{y^2 \geq l_0}\{J(0,u) + J(0,y^2)\}\} \\ &= J(0,u) + J(l_0,0). \end{aligned}$$

Case 1(ii): If $x_2 \leq u$ and $x_1 \leq l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0}\{J(0,u) + J(z_1^2, y^2)\} \\ &= \max\{\max_{l_0 - x_1 \leq y^2 \leq l_0}\{J(0,u) + J(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1}\{J(0,u) + J(x_1, y^2)\} \\ &\quad \max_{y^2 \geq l_0}\{J(0,u) + J(0,y^2)\}\}. \end{aligned}$$

In this case, if $x_1 \le u'$, $K_2(x_1,x_2) = (c-s)x_2 + J(0,u) + J(0,u)$; and if $x_1 \ge u'$, $K_2(x_1,x_2) = (c-s)x_2 + J(0,u) + J(x_1, O(x_1))$.

Case 2(i): If $x_1 \ge l_0$ and $x_2 - u \ge l_0$,

$$\begin{aligned}
K_2 - (c-s)x_2 &= \max\{\max_{0\le z_1^2\le x_1, 0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(z_1^2,y^2)\}, \max_{0\le z_1^2\le x_1, y^2\ge x_2-u}\{J(0,u)+J(z_1^2,y^2)\}\} \\
&= \max\{\max_{0\le y^2\le l_0}\{J(0,x_2-y^2)+J(l_0-y^2,y^2)\}, \max_{l_0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(0,y^2)\}, \\
&\quad \max_{y^2\ge x_2-u}\{J(0,u)+J(0,y^2)\}\}.
\end{aligned}$$

Case 2(ii): If $x_1 \ge l_0$ and $u \le x_2 \le u + l_0$

$$\begin{aligned}
K_2 - (c-s)x_2 &= \max\{\max_{0\le z_1^2\le x_1, 0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(z_1^2,y^2)\}, \max_{0\le z_1^2\le x_1, y^2\ge x_2-u}\{J(0,u)+J(z_1^2,y^2)\}\} \\
&= \max\{\max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(l_0-y^2,y^2)\}, \max_{x_2-u\le y^2\le l_0}\{J(0,u)+J(l_0-y^2,y^2)\}, \\
&\quad \max_{y^2\ge l_0}\{J(0,u)+J(0,y^2)\}\}.
\end{aligned}$$

For case 2(i) and (ii), we can show that if $x_2 \ge u + \tilde{l}$,

$$K_2 - (c-s)x_2 = \max\{\max_{0\le y^2\le \tilde{l}}\{J(0,x_2-y^2)+J(l_0-y^2,y^2)\}, 2J(0,\max\{u,x_2/2\})\};$$

and if $x_2 \le u + \tilde{l}$,

$$K_2 - (c-s)x_2 = \max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(l_0-y^2,y^2)\}.$$

Case 2(iii): If $x_1 \le l_0$ and $l_0 - x_1 \ge x_2 - u \ge 0$,

$$\begin{aligned}
K_2 - (c-s)x_2 &= \max\{\max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(x_1,y^2)\}, \max_{l_0-x_1\le y^2\le l_0}\{J(0,u)+J(l_0-y^2,y^2)\}, \\
&\quad \max_{x_2-u\le y^2\le l_0-x_1}\{J(0,u)+J(x_1,y^2)\}, \max_{y^2\ge l_0}\{J(0,u)+J(0,y^2)\} \\
&= \max\{\max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(x_1,y^2)\}, \max_{l_0-x_1\le y^2\le l_0}\{J(0,u)+J(l_0-y^2,y^2)\}, \\
&\quad \max_{x_2-u\le y^2\le l_0-x_1}\{J(0,u)+J(x_1,y^2)\}\}.
\end{aligned}$$

In this case, if$x_1 \le u'$, $K_2(x_1,x_2) = (c-s)x_2 + J(0,u) + J(0,u)$. If $x_1 \ge u'$ and $x_2 - u \le O(x_1)$, $K_2(x_1,x_2) = (c-s)x_2 + J(0,u) + J(x_1,O(x_1))$. And if $x_1 \ge u'$ and $x_2 - u \ge O(x_1)$,

$$K_2 - (c-s)x_2 = \max\{\max_{0\le y^2\le x_2-u}\{J(0,x_2-y^2)+J(x_1,y^2)\}, J(0,u)+J(0,u)\}.$$

Case 2(iv): If $x_1 \le l_0$ and $l_0 - x_1 \le x_2 - u \le l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{ \max_{l_0 - x_1 \le y^2 \le x_2 - u} \{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1} \{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{x_2 - u \le y^2 \le l_0} \{J(0, u) + J(l_0 - y^2, y^2)\}, \max_{y^2 \ge l_0} \{J(0, u) + J(0, y^2)\} \\ &= \max\{ \max_{l_0 - x_1 \le y^2 \le x_2 - u} \{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1} \{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{x_2 - u \le y^2 \le l_0} \{J(0, u) + J(l_0 - y^2, y^2)\}\}. \end{aligned}$$

Case 2(v): If $x_1 \le l_0$ and $x_2 - u \ge l_0$,

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{ \max_{l_0 - x_1 \le y^2 \le l_0} \{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1} \{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{l_0 \le y^2 \le x_2 - u} \{J(0, x_2 - y^2) + J(0, y^2)\}, \max_{y^2 \ge x_2 - u} \{J(0, u) + J(0, y^2)\}\} \\ &= \max\{ \max_{l_0 - x_1 \le y^2 \le l_0} \{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \max_{0 \le y^2 \le l_0 - x_1} \{J(0, x_2 - y^2) + J(x_1, y^2)\}, \\ &\quad \max_{l_0 \le y^2 \le x_2 - u} \{J(0, x_2 - y^2) + J(0, y^2)\}\}. \end{aligned}$$

For Cases 2(iv) and (v), we can show that if $x_1 \le u'$, we have $K_2 - (c-s)x_2 = 2J(0, \max\{x_2, u\}/2)$; and if $x_1 \ge u'$, we have

$$\begin{aligned} K_2 - (c-s)x_2 &= \max\{ \max_{l_0 - x_1 \le y^2 \le l_0 - \tilde{u}} \{J(0, x_2 - y^2) + J(l_0 - y^2, y^2)\}, \\ &\quad \max_{O(x_1) \le y^2 \le l_0 - x_1} \{J(0, x_2 - y^2) + J(x_1, y^2)\}, 2J(0, \max\{x_2, u\}/2)\}. \end{aligned}$$

Let $C(x_1) = \inf\{x_2 \ge 0 : \bar{z}_1^1(x_1, x_2 | K_1) = 0\}$ and $B(x_1) = \sup\{x_2 \ge 0 : \bar{y}^2(x_1, x_2 | K_2) = 0\}$, then all the results hold. In addition, from the above discussion, we know that if $u' \le l$, $C(x_1)$ must be smaller than

$$\tilde{C}(x_1) = \begin{cases} 0 & \text{if} \quad x_1 \le l; \\ \tilde{l} & \text{if} \quad x_1 \ge l \end{cases}$$

and $B(x_1)$ must be greater than

$$\tilde{B}(x_1) = \begin{cases} 0 & \text{if} \quad x_1 \le l; \\ u & \text{if} \quad x_1 \ge l. \end{cases}$$

If $u' \ge l$, $C(x_1)$ must be smaller than

$$\tilde{C}(x_1) = \begin{cases} 0 & \text{if} \quad x_1 \le u'; \\ \tilde{l} & \text{if} \quad x_1 \ge u' \end{cases}$$

and $B(x_1)$ must be greater than

$$\tilde{B}(x_1) = \begin{cases} 0 & \text{if} \quad x_1 \le u'; \\ u & \text{if} \quad x_1 \ge u'. \end{cases}$$

□

Proof of Theorem 9: The proof consists of two steps.

Step 1. We first prove that if $\bar{z}_1^1 + \bar{y}^1 > \bar{y}^2$, either $\bar{z}_1^1 = 0$ or $\bar{y}^2 = 0$. The proof is by contradiction. Suppose for all optimal policies that satisfy $\bar{z}_1^1 + \bar{y}^1 > \bar{y}^2$, we have $\bar{z}_1^1 > 0$ and $\bar{y}^2 > 0$. Let $\delta = \min(\bar{z}_1^1, \bar{y}^2)$ and we construct a new policy as follows:

$$z_1^1 = \bar{z}_1^1 - \delta,\ z_2^1 = \bar{z}_2^1 + \min(\delta, \bar{z}_2^2),\ y^1 = \bar{y}^1 + \delta,$$
$$z_1^2 = \bar{z}_1^2 + \delta,\ z_2^2 = \bar{z}_2^2 - \min(\delta, \bar{z}_2^2),\ y^2 = \bar{y}^2 - \delta,$$

It is not difficult to show that the new policy is still feasible and either z_1^1 or y^2 is zero and $z_1^1, z_2^1, y^1, z_1^2, z_2^2$, and y^2 are all nonnegative. The objective function J can be written as

$$J(z_1^i, y^i) = -sz_1^i - cy^i + p\mathsf{E}\min(D^i, z_1^i + y^i) - \theta\mathsf{E}(z_1^i + y^i - D^i)^+ + (\theta + \alpha v)\mathsf{E}(y^i - D^i)^+.$$

The expected profit under the new policy minus that under the optimal policy is

$$(\theta + \alpha v)[\mathsf{E}(\bar{y}^1 + \delta - D^1)^+ + \mathsf{E}(\bar{y}^2 - \delta - D^2)^+ - \mathsf{E}(\bar{y}^1 - D^1)^+ - \mathsf{E}(\bar{y}^2 - D^2)^+].$$

We have

$$\begin{aligned} \mathsf{E}(\bar{y}^1 + \delta - D^1)^+ - \mathsf{E}(\bar{y}^1 - D^1)^+ &\geq \mathsf{E}(\bar{y}^2 - D^1)^+ - \mathsf{E}(\bar{y}^2 - \delta - D^1)^+, \\ &\geq \mathsf{E}(\bar{y}^2 - D^2)^+ - \mathsf{E}(\bar{y}^2 - \delta - D^2)^+. \end{aligned}$$

Here the first inequality holds because the function $\mathsf{E}(x - D^1)^+$ is a convex function in x and $\bar{y}^1 + \delta \geq \bar{y}^2$. The second is true because $(x - y)^+$ is submodular in (x, y), and D^2 is larger than D^1 stochastically. Therefore, the new policy achieves a higher profit than the optimal policy, which is a contradiction.

Step 2. We now prove that if $\bar{z}_1^1 + \bar{y}^1 > \bar{y}^2$, then $\bar{z}_1^2 > 0$.

Step 2(i). We first prove that if $\bar{z}_1^1 > 0$, then $\bar{z}_1^1 + \bar{y}^1 \leq \bar{z}_1^2 + \bar{y}^2$. The proof is by contradiction. Suppose in all optimal solutions, $\bar{z}_1^1 + \bar{y}^1 > \bar{z}_1^2 + \bar{y}^2$. Consider a new policy with $z_1^1 = \bar{z}_1^1 - \delta$, $z_1^2 = \bar{z}_1^2 + \delta$, $y^k = \bar{y}^k$ and $z_2^k = \bar{z}_2^k$ for $k = 1, 2$. The new policy is feasible as long as $0 \leq \delta \leq \bar{z}_1^1$.

It is easy to see that the value function under new policy is given by

$$\begin{aligned} & \sum_{i=1}^{2} s(x_i - \bar{z}_i^1 - \bar{z}_i^2) - \sum_{i=1}^{2} c(\bar{y}^i - \bar{z}_2^i) \\ + \quad & p(\bar{z}_1^1 + \bar{y}^1 + \bar{z}_1^2 + \bar{y}^2) \\ - \quad & (p + \theta)[\mathsf{E}(\bar{z}_1^1 + \bar{y}^1 - \delta - D^1)^+ + \mathsf{E}(\bar{z}_1^2 + \bar{y}^2 + \delta - D^2)^+] \\ + \quad & (\theta + \alpha v)[\mathsf{E}(\bar{y}^1 - D^1)^+ + \mathsf{E}(\bar{y}^2 - D^2)^+]. \end{aligned}$$

Its first order derivative with respect to δ is

$$(p+\theta)[\Phi_1(\bar{z}_1^1+\bar{y}^1-\delta)-\Phi_2(\bar{z}_1^2+\bar{y}^2+\delta)].$$

Because $D^2 \geq_{st} D^1$, $\Phi_2(x) \leq \Phi_1(x)$. The above derivative is greater than zero as long as $\bar{z}_1^1+\bar{y}^1-\delta \geq \bar{z}_1^2+\bar{y}^2+\delta$. This means that we can find a new policy that is better than the optimal policy and either $z_1^1=0$ or $z_1^1+y^1=z_1^2+y^2$, which is a contradiction.

Step 2(ii). We shall prove that if $\bar{z}_1^2=0$, then $\bar{y}^1 \leq \bar{y}^2$. The proof is by contradiction. Suppose in all optimal policies, $\bar{y}^1 > \bar{y}^2$. We construct a new policy:

$$z_1^1=\bar{z}_1^1,\ y^1=\bar{y}^1-\delta,\ z_2^1=\bar{z}_2^1-\min\{\delta,\bar{z}_2^1\},$$
$$z_1^2=\bar{z}_1^2,\ y^2=\bar{y}^2+\delta,\ z_2^2=\bar{z}_2^2+\min\{\delta,\bar{z}_2^1\}.$$

It is easy to see that the new policy is feasible as long as $0 \leq \delta \leq \bar{y}^1$. The value function under new policy is given by

$$\begin{aligned} & s(\sum_{i=1}^{2}(x_i-\bar{z}_i^1-\bar{z}_i^2))-c(\sum_{i=1}^{2}(\bar{y}^i-\bar{z}_2^i)) \\ + \quad & p(\bar{z}_1^1+\bar{y}^1+\bar{z}_1^2+\bar{y}^2) \\ - \quad & (p+\theta)[\mathsf{E}(\bar{z}_1^1+\bar{y}^1-\delta-D^1)^+ + \mathsf{E}(\bar{z}_1^2+\bar{y}^2+\delta-D^2)^+] \\ + \quad & (\theta+\alpha v)[\mathsf{E}(\bar{y}^1-\delta-D^1)^+ + \mathsf{E}(\bar{y}^2+\delta-D^2)^+]. \end{aligned}$$

Its first order derivative with respect to δ is

$$\begin{aligned} & -(p+\theta)(\Phi_2(\bar{y}^2+\delta)-\Phi_1(\bar{z}_1^1+\bar{y}^1-\delta))+(\theta+\alpha v)(\Phi_2(\bar{y}^2+\delta)-\Phi_1(\bar{y}^1-\delta)) \\ \geq \quad & -(p-\alpha v)(\Phi_2(\bar{y}^2+\delta)-\Phi_1(\bar{y}^1-\delta)) \\ \geq \quad & 0. \end{aligned}$$

The last inequality holds if $0 \leq \delta \leq (\bar{y}^1-\bar{y}^2)/2$. Therefore, we can find a new policy that is better than the optimal policy and $y^1=y^2$, which is a contradiction.

Step 2(iii). We have already proved that if $\bar{z}_1^2=0$, then $\bar{y}^1 \leq \bar{y}^2$. Now if we further have $\bar{z}_1^1=0$, then $\bar{z}_1^1+\bar{y}^1 \leq \bar{y}^2$. Otherwise if $\bar{z}_1^1>0$, then from Step 2(i), we have $\bar{z}_1^1+\bar{y}^1 \leq \bar{z}_1^2+\bar{y}^2 = \bar{y}^2$. Together, this means that if $\bar{z}_1^2=0$, we have $\bar{z}_1^1+\bar{y}^1 \leq \bar{y}^2$. Equivalently, this means if $\bar{z}_1^1+\bar{y}^1 > \bar{y}^2$, then $\bar{z}_1^2>0$.

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