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The secretary problem with multiple job vacancies and batch candidate arrivals

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ABSTRACT

We extend the secretary problem with multiple vacancies to allow batch arrival of candidates. We establish structural properties of the optimal policies. We show that the optimal reward is convex and submodular in the values of candidates, which means that there is benefit for having a candidate pool with more variable or less interdependent values. Similar properties continue to hold when there are multiple classes of vacancies. Our model is applicable to recruitment, dynamic auctions and sequential investment.

1. Introduction

A recruiter has a fixed number of identical vacancies to fill. The recruiting process consists of a finite number periods. In each period, candidates arrive and are assessed. Assessment outcomes are expressed in terms of scores and they are samples of known distributions. In order to maximize the total scores of accepted candidates, how many offers should the recruiter make in each period?

This problem can be modeled as a large-scale dynamic program and is an extension of the classical secretary problem where there is only one vacancy and only one arrival in each period. Reviews of the secretary problem can be found in [8] and [7]. If the objective is to maximize the cardinal value of the accepted object, the optimal policy of the secretary problem is to accept an object if and only if its value is greater than a threshold.

If the decision maker can accept multiple objects, has an objective of maximizing the total cardinal value of the accepted objects, and the objects may have different weights, then the problem is known as a dynamic stochastic knapsack problem. The knapsack problem has applications in a wide range of industries, including logistics, real estate, finance, restaurant, sports and manufacturing [16]. Papastavrou et al. [16] and Kleywegt and Papastavrou [10] showed that a threshold rule is optimal and they also provided a number of monotonicity and convexity properties. Lin et al. [12] focused on a class of "switch-over" control policies and showed that they are asymptotically optimal. As a special case of the knapsack problem, when all objects have the same weight, the model has its application in dynamic auctions [19].

Our work is also related to the sequential investment problem (see [17] for the model and results and [2] for a recent review). In the knapsack problem, the decision maker rejects or accepts the object in each period. In the sequential investment problem, however, the decision maker determines the investment amount for each opportunity and the return depends on the amount as well as the quality of the opportunity.

In the literature mentioned above, the objects or opportunities are assumed to arrive and to be evaluated one at a time. This, however, is impossible when there are many objects to evaluate and the decision maker can accept more than one object. In a high-volume recruiting environment, for example, to have economy of scale the recruiter should divide the process into multiple periods, each of which corresponds to an application deadline. All candidates arriving in a period will be evaluated and multiple offers be made at once. Because the offer decision in each period is based on the score information of all candidates arriving in that period, as opposed to that of only one candidate, such a batch process can also increase the probability that offers are made to more qualified candidates and hence benefits the recruiter.

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Such an extension is nontrivial. First, when multiple candidates are evaluated at the same time, the state space of the dynamic program must include the scores of all the candidates in the current period and it is multi-dimensional. Second, when candidates are evaluated one at a time, the decision for the recruiter in each period is a binary variable. When candidates arrive and are evaluated in batches, however, the recruiter must decide how many offers to make given the observed scores and the number of offers can be any number between zero and the total number of candidates in that period. In other words, we are facing a dynamic program with discrete variables and the proof of preservation of technical properties after optimization, crucial in the analysis of dynamic programs, requires different tools.

The focus of our study is on the structural properties of the optimal policies. We show that the optimal number of offers to make in each period is decreasing with bounded sensitivity in the number of vacancies that have been filled up to that point. When the scores across periods are independent, the optimal number of offers is increasing with bounded sensitivity with respect to each score. Furthermore, the recruiter's optimal reward is convex and submodular in the scores, which means that the recruiter benefits from having a candidate pool with more variable or less interdependent values. When there are multiple classes of job vacancies with different priority, similar structural results continue to hold.

Although we use recruitment as our motivating example, our model and results are equally applicable to other settings. One example is dynamic auctions. To accommodate bidders' different time preferences, auctions are conducted in multiple periods. Dynamic auctions have been used by airlines to sell air tickets [19], property developers to sell similar apartments in a development, and online auction platforms to sell all kinds of items [9]. In that context, the seller has multiple units of inventory to sell, and the scores represent the bid prices in a period. Another example is sequential investment problem where all opportunities require the same amount of investment, a special case of [17]. Finally, our model may also be applicable to some special cases of resource consumption problems in cloud computing [3].

The main technical tool we use is multimodularity, a concept originally from discrete math and closely related to $L^\natural$ convexity. The tool is valuable in dynamic programs in that it is preserved under optimization and the optimizer of a multimodular function has monotonicity properties with bounded sensitivity. In recent years, there have been many applications of multimodularity or $L^\natural$ convexity in dynamic programs (e.g., [11], [1], [5] for multimodularity, and [20], [15], [6] and many more on $L^\natural$ convexity). Although the tool is well known in the literature, how to use the tool to establish structural properties varies from problem to problem, and in our case, is obvious only in hindsight. In particular, before we can apply the tool of multimodularity, we first need to show various properties of order statistics. These properties are new to the literature.

2. The model

Suppose there are d vacancies to fill and the recruiting process consists of T periods, corresponding to T application deadlines. The total number of candidates arriving in a period is n. Extensions to allow different numbers of candidates in different periods, or incorporating uncertainty in the numbers of candidates are possible. In period t, candidate i's qualification is represented by a random variable S_i^t, which we call a *score*, and is observed before the offer decisions are made. Let s_i be the realization of S_i^t. In each period t, the i-th largest score is denoted by $s_{[i]}$. Let $\mathbf{s} = \{s_1, s_2, \ldots, s_n\}$ and $\mathbf{S}^t = \{S_1^t, S_2^t, \ldots, S_n^t\}$. We assume that scores are independent across periods.

After the completion of the recruitment process, there is an underage cost u for each unfilled vacancy. In each period, the recruiter needs to decide a threshold value and the candidates whose scores are higher than the threshold value will be extended offers to; or equivalently, the recruiter decides the number of candidates to make offers to. Let that number be m. Let q be the total number of candidates who have been extended offers to up to period t. We assume that the total number of offers made can not exceed the recruitment target d; that is, m must satisfy $0 \le m \le d - q$. As such, we can restrict the domain for the state q in the optimal value function $V_t(q, \mathbf{s})$ at period t to be $0 \le q \le d$. The dynamic programming formulation is as follows:

$$V_t(q, \mathbf{s}) = \max_{0 \le m \le \min\{n, d-q\}} J_t(q, m, \mathbf{s}), \tag{1}$$

$$J_t(q, m, \mathbf{s}) = \sum_{i=1}^{m} s_{[i]} + \mathsf{E}V_{t+1}(q + m, \mathbf{S}^{t+1}),$$

and

$$V_{T+1}(q, \mathbf{s}) = -u(d - q).$$

Let $\overline{m}_t(q, \mathbf{s})$ be the optimal policy.

The above model is general and the recruiter may or may not accept exactly d employees in the end, depending on the realized scores, the total number of candidates available, and the value of u. If $nT \ge d$ and the scores are positive, then it is optimal to accept exactly d candidates. The structural properties of the optimal policy are the same regardless.

3. Structural properties

Our first main result is a characterization of the structure of the optimal policy for (1). In each period, a candidate is extended an offer to if and only if his or her score is higher than a threshold value. The optimal number of offers that should be made in a period, $\overline{m}_t(q, \mathbf{s})$, is decreasing in q and the slope is bounded by -1.

Lemma 1.

(i) For $i = 1, \ldots, n$, define

$$\overline{s}_i^t = \mathsf{E}V_{t+1}(q + i - 1, \mathbf{S}^{t+1}) - \mathsf{E}V_{t+1}(q + i, \mathbf{S}^{t+1}). \tag{2}$$

Then $\overline{m}_t = \max\{i \in [1, n] : s_{[i]} \ge \overline{s}_i^t\}$ if the set is non-empty; otherwise $\overline{m}_t = 0$. Furthermore, $\overline{s}_i^t$ is increasing in i and q.

(ii) $-1 \le \overline{m}_t(q + 1, \mathbf{s}) - \overline{m}_t(q, \mathbf{s}) \le 0$.

If $s_{[1]} < \overline{s}_1^t$, we define $\overline{s}^t = +\infty$. Otherwise, we define

$$\overline{s}^t = \max_{1 \le i \le n} \overline{s}_i^t I(s_{[i]} \ge \overline{s}_i^t),$$

where $I(\cdot)$ is the indicator function. When $s_{[i]}$ is *strictly* decreasing in i, because $\overline{s}_i^t$ is increasing in i, in period t, a candidate is extended an offer to if and only if his or her score is higher than the threshold value $\overline{s}^t$. When $s_{[i]}$ is not strictly decreasing in i, it is possible that among the candidates with the same score, some are made offers to, while others are not. Although the optimal policy is easy to describe, it is hard to compute. The values defined in (2) are functions of q as well as $\mathbf{s}$, and to compute them, we must compute the multi-dimensional function V_t inductively. To better understand the structural results, let's look at three special cases. If $n = 1$ (i.e., there is only one candidate in each period), the optimal policy is to accept a candidate in a period if and only if his or her score is higher than a threshold. If $d = 1$, the optimal policy is to accept the best candidate in a period if and only if his or her score is higher than a threshold. If $n = 1$, $d = 1$, and the recruiter must recruit exactly one candidate in the end, the optimal policy is, for $t < T$, to accept the first candidate whose score is higher than a threshold and then the recruiting process stops, and for $t = T$, to accept the last candidate regardless of his or her score. The optimal policy for the last case is well known in the literature (see for example [7]).

Because the optimal number of offers (threshold value) in a period is decreasing (increasing) in the number of candidates whom have been accepted up to that period, it is easy to see that the more total vacancies to fill, the more offers the recruiter should make, or the lower the threshold value in each period.

The main idea used in establishing the structural property of the optimal policy in Lemma 1 is to show that the objective function in the dynamic program, $J_t(q, m, \mathbf{s})$, is anti-multimodular. In what follows, we briefly introduce the concept and relevant properties of multimodularity. The original definition of multimodularity is from [14], but the following definitions and properties are reproduced and adapted from [11].

An n-dimensional set $X \subseteq \mathbb{Z}^n$ is called a *multimodular set* if there exist $\mathbf{a}_i \in \mathbb{Z}^n$ and $b_i \in \mathbb{Z}$ such that $X = \{\mathbf{x} \in \mathbb{Z}^n | \mathbf{a}_i \cdot \mathbf{x} \geq b_i, i = 1, 2, .., m\}$ and $\mathbf{a}_i$ has the form $(0, ..., 0, 1, ..., 1, 0, ..., 0)$; that is, the nonzero components of $\mathbf{a}_i$ are either consecutive 1s or consecutive -1s. The definition of a multimodular set here is the same as a polyhederon satisfying (P1) property in [11].

Let $\mathbf{x} = (x_1, ..., x_n)$. An n-dimensional function $f(\mathbf{x})$ defined on a multimodular set $X \subseteq \mathbb{Z}^n$ is *multimodular* (*anti-multimodular*) if $f(x_1 - z, x_2 - x_1, ..., x_n - x_{n-1})$ is submodular (supermodular) in $(\mathbf{x}, z)$. A one-dimensional function is multimodular (anti-multimodular) if and only if it is discrete convex (concave).

Anti-multimodular functions have some useful properties. If $g(x)$ is a one-dimensional discrete concave function, then $f(\mathbf{x}) = g(x_1 + x_2 + ... + x_n)$ is anti-multimodular in $\mathbf{x}$. The sum of anti-multimodular functions is still anti-multimodular; that is, if $f(\mathbf{x})$ and $g(\mathbf{x})$ are anti-multimodular, then $f(\mathbf{x}) + g(\mathbf{x})$ is anti-multimodular, and if $f(\mathbf{x}, d)$ is anti-multimodular in $\mathbf{x}$ for any given d and D is a random variable, then $\mathsf{E} f(\mathbf{x}, D)$ is anti-multimodular in $\mathbf{x}$.

Anti-multimodularity is preserved under maximization and the maximizer of an anti-multimodular function has monotonicity properties with bounded sensitivity. This property has made it a useful tool in showing structural properties in dynamic programs. To be more specific, if $f(\mathbf{x}, y)$ is an $n+1$ dimensional anti-multimodular function and $\{(\mathbf{x}, y) | \mathbf{x} \in X, y \in Y(\mathbf{x})\}$ is a multimodular set, then

$$g(\mathbf{x}) = \max_{y \in Y(\mathbf{x})} f(\mathbf{x}, y)$$

is anti-multimodular in $\mathbf{x}$. In addition, the optimal solution, denoted by $\overline{y}(\mathbf{x})$, satisfies the following inequalities:

$$-1 \leq \Delta_{x_n} \overline{y} \leq \Delta_{x_{n-1}} \overline{y} \leq ... \leq \Delta_{x_1} \overline{y} \leq 0.$$

Throughout the paper, we use the notation $\Delta_{x_i} f(\mathbf{x})$ to represent $f(\mathbf{x} + \mathbf{e}_i) - f(\mathbf{x})$, where $\mathbf{e}_i$ is the unit vector $(0, ..., 0, 1, 0, ...0)$ whose i-th element is one.

With the tool of multimodularity, we can show that the optimal value function $V_t(q, \mathbf{s})$ and the objective function $J_t(q, m, \mathbf{s})$ in (1) have following properties.

Lemma 2.

(i) The value function $V_t(q, \mathbf{s})$ is discrete concave in q.
(ii) $J_t(q, m, \mathbf{s})$ is anti-multimodular in (q, m).

Proof. The proof is by induction. By definition, $V_{T+1}(q, \mathbf{s})$ is discrete concave in q. Suppose that $V_{t+1}(q, \mathbf{s})$ is discrete concave in q. Then $V_{t+1}(q + m, \mathbf{s})$ is anti-multimodular in (q, m). Because the sum of anti-multimodular functions is anti-multimodular, $\mathsf{E}V_{t+1}(q + m, \mathbf{S}^{t+1}(\mathbf{s}))$ is anti-multimodular in (q, m).

Let $f(\mathbf{s}, m) = \sum_{i=1}^{m} s_{[i]}$. Because $f(\mathbf{s}, m+1) - f(\mathbf{s}, m) = s_{[m+1]}$ and $f(\mathbf{s}, m) - f(\mathbf{s}, m-1) = s_{[m]}$, $f(\mathbf{s}, m)$ is discrete concave in m. Again because the sum of anti-multimodular functions is anti-multimodular, $J_t(q, m, \mathbf{s})$ is anti-multimodular in (q, m). Also, the constraint set is a multimodular set. Therefore, $V_t(q, \mathbf{s})$ is discrete concave in q because anti-multimodularity is preserved under maximization. This completes the induction and the results follow. □

We can now formally state the proof for Lemma 1.

Proof of Lemma 1. Let $\overline{m}(q) = \arg\max_m \{f(q + m) + g(m)\}$ where $f(m)$ and $g(m)$ are discrete concave functions. Since $\overline{m}$ is the optimal solution, we have

$$f(q + m) + g(m) \geq f(q + m - 1) + g(m - 1) \text{ if } m \leq \overline{m}(q),$$

and

$$f(q + m) + g(m) \leq f(q + m - 1) + g(m - 1) \text{ if } m > \overline{m}(q).$$

Then $\overline{m}(q) = \max\{i : g(i) - g(i - 1) \geq f(q + i - 1) - f(q + i)\}$.

(i). Recall that

$$J_t(q, m, \mathbf{s}) = \sum_{i=1}^{m} s_{[i]} + \mathsf{E}V_{t+1}(q + m, \mathbf{S}^{t+1}).$$

Let $g(m) = \sum_{i=1}^{m} s_{[i]}$ and $f(q + m) = \mathsf{E}V_{t+1}(q + m, \mathbf{S}^{t+1}(\mathbf{s}))$. Then $g(\cdot)$ is discrete concave and $f(\cdot)$ is discrete concave (Theorem 2(i)). Then based on our earlier discussion,

$$\begin{aligned} \overline{m}_t &= \max\{i : g(i) - g(i - 1) \geq f(q + i - 1) - f(q + i)\} \\ &= \max\{i : s_{[i]} \geq \mathsf{E}V_{t+1}(q + i - 1, \mathbf{S}^{t+1}) - \mathsf{E}V_{t+1}(q + i, \mathbf{S}^{t+1})\} \\ &= \max\{i : s_{[i]} \geq \overline{s}_i^t\}. \end{aligned}$$

Because $\mathsf{E}V_{t+1}(x, \mathbf{S}^{t+1})$ is discrete concave in x, $\overline{s}_i^t$ is increasing i and q.

(ii). The inequality follows because the value-to-go function $J_t(q, m, \mathbf{s})$ is anti-multimodular in (q, m) (Lemma 2 (ii)). □

We show in the next theorem that the optimal number of offers in a period, $\overline{m}_t(q, \mathbf{s})$, is increasing in $\mathbf{s}$ with a slope bounded by 1. In addition, we discuss the impact of score variability and score interdependency on the recruiter's total reward. For n-dimensional random vectors $\mathbf{X}$ and $\mathbf{X}'$ with finite expectations, we say $\mathbf{X}$ is larger than $\mathbf{X}'$ in *convex order* if $\mathsf{E} f(\mathbf{X}) \geq \mathsf{E} f(\mathbf{X}')$ for any n-dimensional convex function f. We say $\mathbf{X}$ is larger than $\mathbf{X}'$ in *submodular order* if $\mathsf{E} f(\mathbf{X}) \geq \mathsf{E} f(\mathbf{X}')$ for any n-dimensional submodular function f [13]. As a concrete example, for normal random vectors $\mathbf{X}$ and $\mathbf{X}'$ with distributions $N(\mu, \Sigma)$ and $N(\mu', \Sigma')$, $\mathbf{X}$ is larger than $\mathbf{X}'$ in convex order if and only if $\mu = \mu'$ and $\Sigma - \Sigma'$ is nonnegative definite, and $\mathbf{X}$ is larger than $\mathbf{X}'$ in submodular order if and only if $\mathbf{X}$ and $\mathbf{X}'$ have the same marginals and $\sigma_{ij} \leq \sigma'_{ij}$ for all $i, j \in [1, n]$ and $i \neq j$, where σ_{ij} and σ'_{ij} are the covariances in $\mathbf{X}$ and $\mathbf{X}'$, respectively [13].

Theorem 1.

(i) $J_t(q, m, \mathbf{s})$ and $V_t(q, \mathbf{s})$ are convex, increasing and submodular in $\mathbf{s}$.

(ii) $J_t(q,m,\mathbf{s})$ *is supermodular in* (m, s_i).
(iii) $V_t(q,\mathbf{s})$ *is submodular in* (q, s_i).
(iv) $0 \leq \overline{m}_t(q, s_1, ..., s_{i-1}, \tilde{s}_i, s_{i+1}, ..., s_n) - \overline{m}_t(q, s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) \leq 1$ *for any* $\tilde{s}_i > s_i$.
(v) Let $\tilde{V}_t(q,\mathbf{s})$ *denote the optimal value function under score distribution* $\tilde{\mathbf{S}}^t$. *If* $\tilde{\mathbf{S}}^t$ *is larger than* $\mathbf{S}^t$ *in convex order or submodular order for all t, then* $\tilde{V}_t(q,\mathbf{s}) \geq V_t(q,\mathbf{s})$.

The submodularity of the recruiter's total reward means that the scores in the same period are economic substitutes - the marginal value of increasing one score is lower when other scores are higher [18]. As a result, the recruiter may benefit from having a more "diverse" and less interdependent pool of candidates. Also, the recruiter accepts candidates if and only if they are sufficiently qualified in each period, which is similar to exercising an option if and only the price is high. Part (v) of Theorem 1 shows that a higher variability increases the option value.

We introduce the following lemmas, which will be needed in the proof of Theorem 1. Lemma 3 shows that the maximum of linear functions with certain special structures is submodular and will be used in the proof in Lemmas 4 and 5.

Lemma 3. *Let* $f(s_1, s_2) = \max\{s_1 + b, s_2 + b, s_1 + s_2, a + b\}$. *If* $a \leq b$, *then* $f(s_1, s_2)$ *is submodular.*

Proof. Let $\widehat{f}(m_1, m_2) = \max\{m_1 + b, m_2 + b, m_1 + m_2\}$. It is easy to verify that

$$\widehat{f}(m_1, m_2) = \min_{z \in C(m_1, m_2)} \{m_1 + m_2 + b - z\},$$

where the constraint $C(m_1, m_2) = \{z | z \leq m_1, z \leq m_2, z \leq b\}$. Because the objective function is submodular in (m_1, m_2, z) and $C(m_1, m_2)$ is a lattice, $\widehat{f}(m_1, m_2)$ is submodular (Topkis 1998).

When $a \leq b$,

$$f(s_1, s_2) = \min_{(m_1, m_2) \in M(s_1, s_2)} \widehat{f}(m_1, m_2),$$

where the constraint $M(s_1, s_2) = \{(m_1, m_2) | m_1 \geq s_1, m_2 \geq s_2, m_1 \geq a, m_2 \geq a\}$. Similarly, because the objective function $\widehat{f}(m_1, m_2)$ is submodular in (m_1, m_2, s_1, s_2) and $M(s_1, s_2)$ is a lattice, $f(s_1, s_2)$ is submodular. □

The condition $a \leq b$ in Lemma 3 is crucial; without it, the submodularity result may not hold. In the dynamic program (1), the one-period reward function, $\sum_{i=1}^{m} s_{[i]}$, is the sum of the largest m elements of an n-dimensional vector. In Lemma 4, we establish various properties of the reward function. Particularly important is Lemma 4(iii), which states that it is a submodular function.

Lemma 4. *Let* $\mathbf{s} = (s_1, ..., s_n) \in \mathbb{R}^n$ *and* $s_{[i]}$ *be the i-th largest value in* $\mathbf{s}$. *Let* $f(\mathbf{s}, m) = \sum_{i=1}^{m} s_{[i]}$ *where* $1 \leq m \leq n$. *Then*

(i) $f(\mathbf{s}, m)$ *is discrete concave in m.*
(ii) $f(\mathbf{s}, m)$ *is supermodular in* (s_i, m).
(iii) $f(\mathbf{s}, m)$ *is submodular in* $\mathbf{s}$.
(iv) $f(\mathbf{s}, m)$ *is convex increasing in* $\mathbf{s}$.

Proof. (i) Because $f(\mathbf{s}, m+1) - f(\mathbf{s}, m) = s_{[m+1]}$ and $f(\mathbf{s}, m) - f(\mathbf{s}, m-1) = s_{[m]}$, the concavity follows.

(ii). This is because $f(\mathbf{s}, m+1) - f(\mathbf{s}, m) = s_{[m+1]}$, which is increasing s_i.

(iii). We consider two cases. Suppose $m = 1$, then

$$f(\mathbf{s}, 1) = s_{[1]} = \max\{s_1, s_2, ..., s_n\} = \min_{z \in C(\mathbf{s})} z.$$

Here the constraint $C(\mathbf{s}) = \{z | z \geq s_i \text{ for all } i\}$. Because $f(\mathbf{s}, 1)$ is obtained from minimizing a submodular function over a lattice, it is submodular.

Suppose $m \geq 2$. Without loss of generality, we shall just prove that $f(\mathbf{s}, m)$ is submodular in (s_1, s_2). When $n = 2$, the result obviously holds. Hence we only consider $n \geq 3$ in what follows. For $k \in \{m, m-1, m-2\}$, let

$$c(k) = \max\{s_{i_1} + ... + s_{i_k} | 3 \leq i_1 \leq ... \leq i_k \leq n\}.$$

Then $c(k)$ represents the sum of the largest k elements from $\{s_3, ..., s_n\}$. Let $c(0) = 0$. By part (i) of this lemma, we have $c(m) - c(m-1) \leq c(m-1) - c(m-2)$. Then we can rewrite $f(\mathbf{s}, m)$ as

$$f(\mathbf{s}, m) = \max\{s_1 + c(m-1), s_2 + c(m-1), s_1 + s_2 + c(m-2), c(m)\}$$

If we let $a = c(m) - c(m-1)$ and $b = c(m-1) - c(m-2)$, then

$$f(\mathbf{s}, m) = c(m-2) + \max\{s_1 + b, s_2 + b, s_1 + s_2, a + b\}.$$

Because a, b and $c(m-2)$ do not depend on s_1 and s_2, $f(\mathbf{s}, m)$ is submodular in (s_1, s_2) according to Lemma 3. We can similarly prove the submodularity of $f(\mathbf{s}, m)$ with respect to (s_i, s_j) by defining the function $c(k)$ as the sum of the largest k elements from the set $\{s_1, ..., s_n\}$ excluding s_i and s_j.

(iv). For any m, $f(\mathbf{s}, m)$ can be written as

$$f(\mathbf{s}, m) = \sum_{i=1}^{m} s_{[i]} = \max\{s_{i_1} + ... + s_{i_m} | 1 \leq i_1 \leq ... \leq i_m \leq n\}.$$

This shows that $f(\mathbf{s}, m)$ is the point-wise maximum of a finite number of convex functions of $\mathbf{s}$, and hence it is convex in $\mathbf{s}$ [4]. Because $f(\mathbf{s}, m)$ is the maximum of increasing functions of $\mathbf{s}$, it is also increasing in $\mathbf{s}$. □

Lemma 5 is a preservation result showing that convexity and submodularity properties in our model are preserved after maximization, which is a crucial step in showing the structural properties in Theorem 1.

Lemma 5. *Let* $\mathbf{s} = (s_1, ..., s_n) \in \mathbb{R}^n$ *and* $s_{[i]}$ *be the i-th largest value in* $\mathbf{s}$. *Let* $f(\mathbf{s}, m) = \sum_{i=1}^{m} s_{[i]}$ *where* $1 \leq m \leq n$. *Let* $f(\mathbf{s}, 0) = 0$ *and* $g(m)$ *be any function of m. Let*

$$h(\mathbf{s}) = \max_{0 \leq m \leq n} \{f(\mathbf{s}, m) + g(m)\}.$$

(i) The function $h(\mathbf{s})$ *is convex increasing in* $\mathbf{s}$.
(ii) If $g(m)$ *is discrete concave, then* $h(\mathbf{s})$ *is submodular in* $\mathbf{s}$.

Proof. (i). For any given integer $m \in [1, n]$, $f(\mathbf{s}, m)$ is convex in $\mathbf{s}$ (Lemma 4), therefore $f(\mathbf{s}, m) + g(m)$ is convex in $\mathbf{s}$. And

$$h(\mathbf{s}) = \max\{f(\mathbf{s}, 0) + g(0), f(\mathbf{s}, 1) + g(1), ..., f(\mathbf{s}, n) + g(n)\},$$

which is the maximum of $n+1$ convex functions of $\mathbf{s}$. So it is also convex in $\mathbf{s}$. Because $h(\mathbf{s})$ is the maximum of increasing functions of $\mathbf{s}$, it is also increasing in $\mathbf{s}$.

(ii). Without loss of generality, we shall prove that $h(\mathbf{s})$ is submodular in (s_1, s_2). From the proof of Lemma 4 (iii), we know that if $m \geq 2$,

$$f(\mathbf{s}, m) = \max\{s_1 + c(m-1), s_2 + c(m-1), s_1 + s_2 + c(m-2), c(m)\}.$$

Let

$$\tilde{c}(m) = \max_{m \le i \le n} \{c(i-m) + g(i)\}.$$

Let $i = i' + m$, then

$$\tilde{c}(m) = \max_{m \le i'+m \le n} \{c(i') + g(i'+m)\}.$$

If $g(\cdot)$ is discrete concave, then the objective function $c(i') + g(i' + m)$ is anti-multimodular in (i', m). Because the set $\{(i', m) | m \le i' + m \le n\}$ is a multimodular set, $\tilde{c}(m)$ is discrete concave in m.

By the definition of $\tilde{c}(m)$, we can rewrite $h(\mathbf{s})$ as

$$\begin{aligned} h(\mathbf{s}) &= \max\{s_1 + \tilde{c}(1), s_2 + \tilde{c}(1), s_1 + s_2 + \tilde{c}(2), \tilde{c}(0)\} \\ &= \tilde{c}(2) + \max\{s_1 + \tilde{c}(1) - \tilde{c}(2), s_2 + \tilde{c}(1) - \tilde{c}(2), \\ &\qquad s_1 + s_2, \tilde{c}(0) - \tilde{c}(1) + \tilde{c}(1) - \tilde{c}(2)\}. \end{aligned}$$

Because $\tilde{c}(m)$ is discrete concave, we have $\tilde{c}(0) - \tilde{c}(1) \le \tilde{c}(1) - \tilde{c}(2)$, and from Lemma 3, we can conclude that $h(\mathbf{s})$ is submodular in (s_1, s_2). □

We can now state the proof for Theorem 1.

Proof of Theorem 1. (i). $J_t(q, m, \mathbf{s})$ and $V_t(q, \mathbf{s})$ are convex increasing in $\mathbf{s}$ by Lemma 4(iv) and Lemma 5(i). The submodularity comes from Lemma 4(iii) and Lemma 5(ii).

(ii). The result follows directly from Lemma 4(ii).

(iii) Let $\tilde{q} = -q$. Then we have

$$V_t(-\tilde{q}, \mathbf{s}) = \max_{0 \le m \le n} J_t(-\tilde{q}, m, \mathbf{s}).$$

Because $J_t(-\tilde{q}, m, \mathbf{s})$ is supermodular in $(\tilde{q}, m, s_i)$, $V_t(-\tilde{q}, \mathbf{s})$ is supermodular in $(\tilde{q}, s_i)$. Therefore, $V_t(q, \mathbf{s})$ is submodular in (q, s_i).

(iv). Because $J_t(q, m, \mathbf{s})$ is supermodular in (m, s_i), $\overline{m}_t(q, \mathbf{s})$ is increasing in s_i. From Theorem 1(i), we have $\bar{s}_i^t = \mathsf{E}V_{t+1}(q + i - 1, \mathbf{S}^{t+1}) - \mathsf{E}V_{t+1}(q + i, \mathbf{S}^{t+1})$ and

$$\overline{m}_t(q, \mathbf{s}) = \max\{i : s_{[i]} \ge \bar{s}_i^t\}.$$

Note that $\bar{s}_i^t$ does not depend on $\mathbf{s}$. Suppose that candidate i's score is increased from s_i to $\tilde{s}_i$. If the candidate was among the top $\overline{m}_t$ candidates before the change of score, then the optimal number of offers is still equal to $\overline{m}_t$.

If candidate i was not among the top $\overline{m}_t$ candidates before the change of score, then there are two cases to consider.

Case 1. Suppose that candidate i is now among the top $\overline{m}_t$ candidates. Then the previous top $\overline{m}_t - 1$ candidates will still be accepted. This is because if $i < \overline{m}_t$, then $s_{[i]} \ge s_{[i+1]} \ge \bar{s}_{i+1}$. If the previous $\overline{m}_t$-th highest score is smaller than $\bar{s}_{\overline{m}_t+1}^t$, then the optimal number of offers is $\overline{m}_t$; otherwise the optimal number of offers is $\overline{m}_t + 1$.

Case 2. Suppose that candidate i is not among the top $\overline{m}_t$ candidates. If candidate i's score is the $(\overline{m}_t + 1)$-th highest and $\tilde{s}_i^t \ge \bar{s}_{\overline{m}_t+1}^t$, then the optimal number of offers is $\overline{m}_t + 1$; otherwise the optimal number of offers is still $\overline{m}_t$. In summary, the optimal number of offers cannot increase more than one when one score increases, regardless of the amount.

(v). The proof is by induction on t. By definition, $\tilde{V}_{T+1}(q, \mathbf{s}) \ge V_{T+1}(q, \mathbf{s})$. Suppose $\tilde{V}_{t+1}(q, \mathbf{s}) \ge V_{t+1}(q, \mathbf{s})$. Then

$$\begin{aligned} \tilde{J}_t(q, m, \mathbf{s}) &= \sum_{i=1}^{m} s_{[i]} + \mathsf{E}\tilde{V}_{t+1}(q + m, \tilde{\mathbf{S}}^{t+1}) \\ &\ge \sum_{i=1}^{m} s_{[i]} + \mathsf{E}\tilde{V}_{t+1}(q + m, \mathbf{S}^{t+1}) \\ &\ge \sum_{i=1}^{m} s_{[i]} + \mathsf{E}V_{t+1}(q + m, \mathbf{S}^{t+1}) \\ &= J_t(q, m, \mathbf{s}). \end{aligned}$$

Because $\tilde{V}_{t+1}(q + m, \mathbf{s})$ is convex and submodular in $\mathbf{s}$, the first inequality holds if $\tilde{\mathbf{S}}^{t+1}$ is larger than $\mathbf{S}^{t+1}$ in convex order or submodular order. The second inequality is due to the induction hypothesis. Therefore, $\tilde{V}_t(q, \mathbf{s}) \ge V_t(q, \mathbf{s})$. The induction is complete. □

The next result shows that if the score vector is i.i.d across periods, then the recruiter should make more offers when it is getting closer to the end of the planning horizon.

Theorem 2. *If the score vector $\mathbf{S}^t$ is i.i.d across periods, then $\overline{m}_t(q, \mathbf{s})$ is increasing in t.*

Proof. From Lemma 1(i) we have $\bar{s}_i^t = \mathsf{E}V_{t+1}(q + i - 1, \mathbf{S}) - \mathsf{E}V_{t+1}(q + i, \mathbf{S})$ and

$$\overline{m}_t(q, \mathbf{s}) = \max\{i : s_{[i]} \ge \bar{s}_i^t\}.$$

To prove the result, it suffices to show that $\bar{s}_i^t$ is decreasing in t, or

$$\mathsf{E}V_t(q, \mathbf{S}) - \mathsf{E}V_t(q-1, \mathbf{S}) \le \mathsf{E}V_{t+1}(q, \mathbf{S}) - \mathsf{E}V_{t+1}(q-1, \mathbf{S}). \quad (3)$$

To simplify the presentation, denote $\overline{m}_t(q, \mathbf{s})$ by $\overline{m}$. According to Lemma 1(ii), either $\overline{m}_t(q-1, \mathbf{s})$ is equal to $\overline{m}$ or $\overline{m} + 1$. Thus we have

$$\begin{aligned} V_t(q-1, \mathbf{s}) = \max\{&\sum_{i=1}^{\overline{m}} s_{[i]} + \mathsf{E}V_{t+1}(q + \overline{m} - 1, \mathbf{S}), \\ &\sum_{i=1}^{\overline{m}+1} s_{[i]} + \mathsf{E}V_{t+1}(q + \overline{m}, \mathbf{S})\} \end{aligned}$$

and

$$V_t(q, \mathbf{s}) = \sum_{i=1}^{\overline{m}} s_{[i]} + \mathsf{E}V_{t+1}(q + \overline{m}, \mathbf{S}).$$

Therefore,

$$\begin{aligned} V_t(q, \mathbf{s}) - V_t(q-1, \mathbf{s}) &\le \mathsf{E}V_{t+1}(q + \overline{m}, \mathbf{S}) - \mathsf{E}V_{t+1}(q + \overline{m} - 1, \mathbf{S}) \\ &\le \mathsf{E}V_{t+1}(q, \mathbf{S}) - \mathsf{E}V_{t+1}(q-1, \mathbf{S}). \end{aligned}$$

The second inequality is due to the concavity of $\mathsf{E}V_{t+1}(q, \mathbf{S})$ in q. Because the above inequality holds for any $\mathbf{s}$, inequality (3) must also hold and the result hence follows. □

4. Conclusions

The assumption that the scores are independent across periods is needed for Theorem 1, but not for Lemmas 1 and 2. Lemmas 3 and 5 involve showing the submodularity of a function which is the parameterized maximum of another function. Lemma 4 provides properties of the sum of order statistics. These lemmas pave the way for establishing the structural properties in this work and are nonstandard. We hope that they will be useful elsewhere.

In our earlier analysis, job vacancies are identical. Similar structural properties continue to hold when there are multiple classes of vacancies. Like in the single-class case, there is a threshold value in each period such that a candidate is accepted if and only if his or her score is higher than the threshold value. Accepted candidates are assigned to classes with higher priority until their hiring targets have been met before they are assigned to the classes with lower priority. The threshold value in each period is decreasing in the number of candidates who have already been accepted in any class, and it is more sensitive to the changes in the numbers in the classes with higher priority. The details are provided in the Appendix.

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Appendix A

Suppose there are k classes of job vacancies that share the same pool of candidates. Let d_i denote the recruitment target for class i. Job vacancies have different priorities in the following sense. At the end of the recruitment process, the underage cost for each unfilled vacancy of class i is u_i and $u_1 \geq u_2 \geq \ldots \geq u_k$. Therefore, leaving vacancies unfilled in a job class with a smaller index incurs a higher cost than in a job class with a larger index.

In period t, let q_i denote the number of candidates who have been accepted and assigned to job class i and $\mathbf{q} = (q_1, \ldots, q_k)$. The recruiter must decide the number of offers to make, denoted by m_i, for each class i. We assume that for all classes, the recruiter will not accept more than the targets regardless of the scores; that is, m_i must satisfy $0 \leq m_i \leq d_i - q_i$. As such, we can restrict the domain for the state $\mathbf{q}$ in the value function $V_t(\mathbf{q}, \mathbf{s})$ to be $\mathbf{0} \leq \mathbf{q} \leq \mathbf{d}$ where $\mathbf{d} = (d_1, \ldots, d_k)$. Let $\mathbf{m} = (m_1, \ldots, m_k)$. The dynamic program is as follows.

$$V_t(\mathbf{q}, \mathbf{s}) = \max_{\mathbf{m} \in O} J_t(\mathbf{q}, \mathbf{m}, \mathbf{s}), \tag{A.1}$$

$$J_t(\mathbf{q}, \mathbf{m}, \mathbf{s}) = \sum_{j=1}^{m_1+\ldots+m_k} s_{[j]} + \mathsf{E}V_{t+1}(q_1 + m_1, \ldots, q_k + m_k, \mathbf{S}^{t+1}).$$

Here, the constraint set $O = \{\mathbf{m} : \sum_{j=1}^{k} m_j \leq n, 0 \leq m_i \leq d_i - q_i \ \textit{for all}\ i = 1, \ldots, n\}$. Recall that n is the number of candidates in each period. The terminal condition is $V_{T+1}(\mathbf{q}, \mathbf{s}) = \sum_{i=1}^{k} -u_i(d_i - q_i)$. Let $\overline{\mathbf{m}}^t = (\overline{m}_1^t, \ldots, \overline{m}_i^t, \ldots, \overline{m}_k^t)$ denote the optimal solution. The structural properties of the optimal policy is provided in Theorem A.1.

Theorem A.1.

(i) The optimal number of offers for job class i satisfies the following inequalities.

$$-1 \leq \Delta_{q_i} \overline{m}_i^t \leq \Delta_{q_{i+1}} \overline{m}_i^t \leq \ldots \leq \Delta_{q_k} \overline{m}_i^t \leq 0 \tag{A.2}$$

and

$$0 \leq \Delta_{q_1} \overline{m}_i^t = \Delta_{q_2} \overline{m}_i^t = \ldots = \Delta_{q_{i-1}} \overline{m}_i^t \leq 1. \tag{A.3}$$

(ii) Let $\overline{m}^t = \sum_{j=1}^{k} \overline{m}_j^t$ be the optimal number of total offers in period t. The following inequalities hold:

$$-1 \leq \Delta_{q_1} \overline{m}^t \leq \Delta_{q_2} \overline{m}^t \leq \ldots \leq \Delta_{q_k} \overline{m}^t \leq 0.$$

(iii) In period t, there exists a threshold $\overline{s}_t(\mathbf{q}, \mathbf{s})$ such that a candidate is accepted if and only if his or her score is above $\overline{s}_t(\mathbf{q}, \mathbf{s})$. Furthermore,

$$\Delta_{q_1} \overline{s}_t \geq \Delta_{q_2} \overline{s}_t \geq \ldots \geq \Delta_{q_k} \overline{s}_t \geq 0.$$

Theorem A.1 is an extension of Lemma 1. Similar to the case with only one class, there is also a threshold value in each period such that a candidate is accepted if and only if his or her score is higher than the threshold value. Besides the results about the signs and the bounds of the derivatives that Lemma 1 shows, Theorem A.1 also shows that the optimal number of total offers and the threshold are more sensitive to the changes in the numbers of accepted candidates in classes with higher priority than those with lower priority. In addition, the optimal number of offers for job class i is decreasing in the number of accepted candidates in class i or any class with lower priority, but increasing in that in any class with a higher priority than i.

The proof of Theorem A.1 requires properties about the value function $V_t(\mathbf{q}, \mathbf{s})$ with respect to $\mathbf{q}$. These properties are provided in Theorem A.2.

Theorem A.2.

(i) $\Delta_{q_1} V_t(\mathbf{q}, \mathbf{s}) \geq \Delta_{q_2} V_t(\mathbf{q}, \mathbf{s}) \geq \ldots \geq \Delta_{q_k} V_t(\mathbf{q}, \mathbf{s})$;
(ii) $V_t(\mathbf{q}, \mathbf{s})$ is anti-multimodular in $\mathbf{q}$.

Proof. (i). The proof is by induction. In period $T + 1$, the result holds because $\Delta_{q_i} V_t(\mathbf{q}, \mathbf{s}) = u_i$ and $u_i \geq u_{i+1}$. Suppose the result holds for period $t + 1$. That is, $\Delta_{q_i} V_{t+1}(\mathbf{q}, \mathbf{s}) \geq \Delta_{q_{i+1}} V_{t+1}(\mathbf{q}, \mathbf{s})$. Then it is easy to verify that $\Delta_{m_i} J_t(\mathbf{q}, \mathbf{m}, \mathbf{s}) \geq \Delta_{m_{i+1}} J_t(\mathbf{q}, \mathbf{m}, \mathbf{s})$. This means that it is optimal to fill positions with higher priorities before lower ones. Let $\overline{\mathbf{m}}$ be the optimal solution for the state $(\mathbf{q}, \mathbf{s})$. Then either $\overline{m}_i = d_i - q_i$ or $\overline{m}_{i+1} = 0$.

If $\overline{m}_i = d_i - q_i$, then $\overline{\mathbf{m}} - \mathbf{e}_i + \mathbf{e}_{i+1}$ is a feasible solution for the state $(\mathbf{q} + \mathbf{e}_i - \mathbf{e}_{i+1}, \mathbf{s})$. Then

$$\begin{aligned} &V_t(\mathbf{q} + \mathbf{e}_i - \mathbf{e}_{i+1}, \mathbf{s}) - V_t(\mathbf{q}, \mathbf{s}) \\ &\geq \mathsf{E}V_{t+1}(\ldots, q_i + 1 + \overline{m}_i - 1, q_{i+1} - 1 + \overline{m}_{i+1} + 1, \ldots, \mathbf{S}^{t+1}(\mathbf{s})) \\ &\quad - \mathsf{E}V_{t+1}(\ldots, q_i + \overline{m}_i, q_{i+1} + \overline{m}_{i+1}, \ldots, \mathbf{S}^{t+1}(\mathbf{s})), \end{aligned}$$

which is equal to zero. If $\overline{m}_i < d_i - q_i$, then $\overline{m}_{i+1} = 0$. In this case, $\overline{\mathbf{m}}$ is a feasible solution for the state $(\mathbf{q} + \mathbf{e}_i - \mathbf{e}_{i+1}, \mathbf{s})$, then

$$\begin{aligned} &V_t(\mathbf{q} + \mathbf{e}_i - \mathbf{e}_{i+1}, \mathbf{s}) - V_t(\mathbf{q}, \mathbf{s}) \\ &\geq \mathsf{E}V_{t+1}(\ldots, q_i + 1 + \overline{m}_i, q_{i+1} + \overline{m}_{i+1} - 1, \ldots, \mathbf{S}^{t+1}(\mathbf{s})) \\ &\quad - \mathsf{E}V_{t+1}(\ldots, q_i + \overline{m}_i, q_{i+1} + \overline{m}_{i+1}, \ldots, \mathbf{S}^{t+1}(\mathbf{s})), \end{aligned}$$

which is non-negative. Therefore, $\Delta_{q_i} V_t(\mathbf{q}, \mathbf{s}) \geq \Delta_{q_{i+1}} V_t(\mathbf{q}, \mathbf{s})$. This completes the induction.

(ii). We define the set T by:

$$T = \{\mathbf{m} : 0 \leq \sum_{j=1}^{k} m_j \leq n, 0 \leq \sum_{j=i}^{k} m_j \leq \sum_{j=i}^{k} (d_j - q_j) \ \textit{for all}\ i = 1, \ldots, n\}.$$

Because it is optimal to fill vacancies in a class with higher priority as much as possible before one with lower priority, it is easy to show that if we change the constraint set O to constraint set T in (A.1), the optimal solution to the dynamic program (A.1) remains the same.

Let $\tilde{m}_i = -(q_i + m_i)$, $\tilde{\mathbf{m}} = (\tilde{m}_k, \ldots, \tilde{m}_1)$,

$$\tilde{J}_t(\mathbf{q}, \tilde{\mathbf{m}}, \mathbf{s}) = \sum_{j=1}^{-q_1 - \tilde{m}_1 - \ldots - q_k - \tilde{m}_k} s_{[j]} + \mathsf{E}V_{t+1}(-\tilde{m}_1, \ldots, -\tilde{m}_k, \mathbf{S}^{t+1}(\mathbf{s})),$$

and

$$\tilde{T}=\{\tilde{\mathbf{m}}:0\le\sum_{j=1}^{k}-\tilde{m}_j-q_j\le n,0\le\sum_{j=i}^{k}-\tilde{m}_j-q_j\le\sum_{j=i}^{k}d_j-q_j \text{ for all } i\}.$$

Then the dynamic program (A.1) is equivalent to $V_t(\mathbf{q},\mathbf{s})=\max_{\tilde{\mathbf{m}}\in\tilde{T}}\tilde{J}_t(\mathbf{q},\tilde{\mathbf{m}},\mathbf{s})$. The rest of the proof is by induction arguments. Obviously, $V_{T+1}(\mathbf{q},\mathbf{s})$ is anti-multimodular in $\mathbf{q}$. Suppose $V_{t+1}(\mathbf{q},\mathbf{s})$ is anti-multimodular in $\mathbf{q}$. Then by definition of anti-multimodularity, we can verify that $\tilde{J}_t(\mathbf{q},\tilde{\mathbf{m}},\mathbf{s})$ is anti-multimodular in $(\mathbf{q},\tilde{\mathbf{m}})$ and the constraint set $\tilde{T}$ forms a multimodular set. Therefore, $V_t(\mathbf{q},\mathbf{s})$ is anti-multimodular in $\mathbf{q}$. □

Based on Theorem A.2 (i), the marginal values of accepted candidates follow the same order as their respective classes; that is, the higher the priority of a class, the higher the marginal value of its accepted candidates. Theorem A.2 (ii) means that the accepted candidates in different classes are economic substitutes; the marginal value of them in one class increases if the number in a different class decreases. With these properties, we can now present the proof for Theorem A.1.

Proof of Theorem A.1. (i) Let $\tilde{q}_i=\sum_{j=1}^{i-1}d_j-q_j$. From the proof of Theorem A.2, $\overline{m}_i^t$ is the solution to the following one-dimensional optimization problem.

$$\overline{m}_i^t(\mathbf{q},\mathbf{s})=\arg\max_{0\le m_i\le\min\{n-\tilde{q}_i,d_i-q_i\}}\{\sum_{j=1}^{\tilde{q}_i+m_i}s_{[j]}+\mathsf{E}V_{t+1}(d_1,\dots,d_{i-1},q_i+m_i,q_{i+1},\dots,q_k,\mathbf{S}^{t+1})\}.$$

We can verify that the objective function above is anti-multimodular in $(m_i,q_i,\dots,q_k)$, which leads to (A.2). It is also anti-multimodular in $(m_i,\tilde{q}_i)$, which leads to (A.3).

(ii) It is obvious from (A.2) and (A.3) that $\Delta_{q_k}\overline{m}^t\le0$ and $\Delta_{q_1}\overline{m}^t\ge-1$. We shall prove $\Delta_{q_i}\overline{m}^t\le\Delta_{q_{i+1}}\overline{m}^t$ for $i=1,\dots,k-1$. We first show that the following inequality holds.

$$\Delta_{q_i}\overline{m}_i^t+\Delta_{q_i}\overline{m}_{i+1}^t\le\Delta_{q_{i+1}}\overline{m}_i^t+\Delta_{q_{i+1}}\overline{m}_{i+1}^t. \tag{A.4}$$

To prove this, notice that $(\overline{m}_i^t,\overline{m}_{i+1}^t)$ is the solution to the following optimization problem.

$$\max_{(m_i,m_{i+1})\in M}f(q_i,q_{i+1},m_i,m_{i+1})$$

where $M=\{(m_i,m_{i+1}):0\le m_i+m_{i+1}\le\min\{n-\tilde{q}_i,d_i-q_i+d_{i+1}-q_{i+1}\},m_{i+1}\le d_{i+1}-q_{i+1}\}$,

$$f(q_i,q_{i+1},m_{i+1},m_i)=\sum_{j=1}^{\tilde{q}_i+m_i+m_{i+1}}s_{[j]}+\mathsf{E}V_{t+1}(d_1,\dots,d_{i-1},q_i+m_i,q_{i+1}+m_{i+1},q_{i+2},\dots,q_k,\mathbf{S}^{t+1}).$$

Using a similar idea used in the proof of Theorem A.2, we can show that the function

$$\tilde{f}(q_i,q_{i+1},m'_{i+1},m'_i)=f(q_i,q_{i+1},-m'_{i+1}-q_{i+1},-m'_i-q_i)$$

is anti-multimodular in $(q_i,q_{i+1},m'_{i+1},m'_i)$ where $m'_i=-m_i-q_i$ and $m'_{i+1}=-m_{i+1}-q_{i+1}$.

Let $(\overline{m}'_i,\overline{m}'_{i+1})=\arg\max_{(m'_i,m'_{i+1})\in M'}\tilde{f}(q_i,q_{i+1},m'_{i+1},m'_i)$, where $M'=\{(m'_i,m'_{i+1}):0\le-m'_i-m'_{i+1}\le\min\{n-\tilde{q}_i+q_i+q_{i+1},d_i+d_{i+1}\},-m'_{i+1}\le d_{i+1}\}$. Then $\overline{m}'_i=-\overline{m}_i^t-q_i$ and $\overline{m}'_{i+1}=-\overline{m}_{i+1}^t-q_{i+1}$. Fix m'_{i+1}, let $\tilde{m}'_i(q_i,q_{i+1},m'_{i+1})$ denote the optimal solution from optimizing over m'_i of $\tilde{f}(q_i,q_{i+1},m'_{i+1},m'_i)$, that is,

$$\tilde{m}'_i(q_i,q_{i+1},m'_{i+1})=\arg\max_{\substack{m'_{i+1}+q_i+q_{i+1}\le-m'_i\le m'_{i+1}+q_i+q_{i+1}+n-\tilde{q}_i,\\ -m'_i\le m'_{i+1}+d_i+d_{i+1}}}\tilde{f}(q_i,q_{i+1},m'_{i+1},m'_i).$$

Then we have

$$-1\le\Delta_{m'_{i+1}}\tilde{m}'_i(q_i,q_{i+1},m'_{i+1})\le\Delta_{q_{i+1}}\tilde{m}'_i(q_i,q_{i+1},m'_{i+1})\le\Delta_{q_i}\tilde{m}'_i(q_i,q_{i+1},m'_{i+1})\le0$$

and

$$-1\le\Delta_{q_{i+1}}\overline{m}'_{i+1}\le\Delta_{q_i}\overline{m}'_{i+1}\le0.$$

Because $\overline{m}'_i=\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})$, we have

$$\Delta_{q_i}\overline{m}'_i=\Delta_{q_i}\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})+\Delta_{m'_{i+1}}\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})\Delta_{q_i}\overline{m}'_{i+1}.$$

Thus, $\Delta_{q_i}\overline{m}'_i+\Delta_{q_i}\overline{m}'_{i+1}=\Delta_{q_i}\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})+(\Delta_{m'_{i+1}}\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})+1)\Delta_{q_i}\overline{m}'_{i+1}$. Now that we have $\Delta_{q_i}\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})\ge\Delta_{q_{i+1}}\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})$, $\Delta_{m'_{i+1}}\tilde{m}'_i(q_i,q_{i+1},\overline{m}'_{i+1})+1\ge0$ and $\Delta_{q_i}\overline{m}'_{i+1}\ge\Delta_{q_{i+1}}\overline{m}'_{i+1}$, the following inequality holds:

$$\Delta_{q_i}\overline{m}'_i+\Delta_{q_i}\overline{m}'_{i+1}\ge\Delta_{q_{i+1}}\overline{m}'_i+\Delta_{q_{i+1}}\overline{m}'_{i+1}.$$

This is equivalent to $-1-\Delta_{q_i}\overline{m}_i^t-\Delta_{q_i}\overline{m}_{i+1}^t\ge-1-\Delta_{q_{i+1}}\overline{m}_i^t-\Delta_{q_{i+1}}\overline{m}_{i+1}^t$, hence the inequality (A.4) holds. Then we have

$$\begin{aligned}\Delta_{q_i}\overline{m}^t&=\sum_{j=1}^{i-1}\Delta_{q_i}\overline{m}_j^t+\Delta_{q_i}\overline{m}_i^t+\Delta_{q_i}\overline{m}_{i+1}^t+\sum_{j=i+2}^{k}\Delta_{q_i}\overline{m}_j^t\\&=\sum_{j=1}^{i-1}\Delta_{q_i}\overline{m}_j^t+\Delta_{q_i}\overline{m}_i^t+\Delta_{q_i}\overline{m}_{i+1}^t+\sum_{j=i+2}^{k}\Delta_{q_{i+1}}\overline{m}_j^t\\&\le\sum_{j=1}^{i-1}\Delta_{q_{i+1}}\overline{m}_j^t+\Delta_{q_i}\overline{m}_i^t+\Delta_{q_i}\overline{m}_{i+1}^t+\sum_{j=i+2}^{k}\Delta_{q_{i+1}}\overline{m}_j^t\\&\le\sum_{j=1}^{i-1}\Delta_{q_{i+1}}\overline{m}_j^t+\Delta_{q_{i+1}}\overline{m}_i^t+\Delta_{q_{i+1}}\overline{m}_{i+1}^t+\sum_{j=i+2}^{k}\Delta_{q_{i+1}}\overline{m}_j^t\\&=\Delta_{q_{i+1}}\overline{m}^t.\end{aligned}$$

The second equality is due to (A.3) and the first inequality is due to (A.2). The second inequality is due to (A.4).

(iii) Define the threshold $\overline{s}_t(\mathbf{q},\mathbf{s})$ as the score of the candidate with the $\overline{m}^t$-th highest score. The result follows because $\Delta_{q_i}\overline{s}_t\ge\Delta_{q_{i+1}}\overline{s}_t$ is equivalent to $\Delta_{q_i}\overline{m}^t\le\Delta_{q_{i+1}}\overline{m}^t$. □

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